§ 4.3  The Equations of Compatibility and
The Gauss Theorem

Similar to Frenet frame \( \{ t, n, b \} \) for a curve,
we consider the moving frame \( \{ X_u, X_v, N \} \) for a surface.

**Orthogonal only if** \( F = \langle X_u, X_v \rangle \equiv 0 \)**

Let \( (u_1, u_2) = (u, v) \), \( F = u, v \), \( \partial u_i X = x_s i \); index after \( \partial \) means derivative

\[
x_{ij} = \partial u_i \partial u_j X = \Gamma_{ij}^k x_k + \Gamma_{ij}^2 x_v + \Pi_{ij} N
\]

\[
N_{ij} = a_{1i} x_u + a_{2i} x_v
\]

Weingarten eq

\[
\begin{pmatrix}
\Pi_{ij} \\
\gamma_{1i} \\
\gamma_{2i}
\end{pmatrix}
= \begin{pmatrix}
\langle X_{ij}, N \rangle \\
\langle x_{1i}, x_u \rangle \\
\langle x_{2i}, x_v \rangle
\end{pmatrix}
= \begin{pmatrix}
e & f \\
f & g
\end{pmatrix}
\]

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
= -\begin{pmatrix}
EF & f \\
FG & g
\end{pmatrix}
= \frac{1}{EG-F^2}
\begin{pmatrix}
eE - fG & gE - fF \\
fE - fG & gE - fF
\end{pmatrix}
\]

\[
\Gamma_{ij}^{k} = \Gamma_{ji}^{k}
\]

Christoffel symbols 8 of them

Since \( X_{uv} = X_{vu} \),

\[
\Gamma_{12}^k x_u + \Gamma_{22}^k x_v = \Gamma_{21}^k x_u + \Gamma_{11}^k x_v
\]

\[
\Gamma_{ij}^{k} = \Gamma_{ji}^{k}
\]

we can solve them from the system of \( \langle X_{ij}, X_k \rangle \), e.g.

\[
\langle X_{uv}, X_u \rangle = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v = \frac{1}{2} E_u
\]

\[
\langle X_{uv}, X_v \rangle = \Gamma_{11}^1 x_v + \Gamma_{11}^2 x_u = d_u \langle X_u, X_v \rangle - \langle X_u, X_{uv} \rangle
\]

\[
= F_u - \frac{1}{2} E_v
\]

\[
\begin{pmatrix}
\Gamma_{11}^1 \\
\Gamma_{11}^2
\end{pmatrix}
= \begin{pmatrix}
E & F \\
G & G
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} E_u \\
F_u - \frac{1}{2} E_v
\end{pmatrix}
\]

Similarly

\[
\begin{pmatrix}
\Gamma_{12}^1 \\
\Gamma_{12}^2
\end{pmatrix}
= \begin{pmatrix}
E & F \\
G & G
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} E_v \\
F - \frac{1}{2} G_u
\end{pmatrix}
\]

\[
\begin{pmatrix}
\Gamma_{22}^1 \\
\Gamma_{22}^2
\end{pmatrix}
= \begin{pmatrix}
E & F \\
G & G
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} G_v \\
\frac{1}{2} G_u
\end{pmatrix}
\]
Compatibility equations:

\[(X_{uv})_v = (X_{uv})_u\]  
\[3 \text{ eq. fn coeff.}\]

\[(X_{uv})_u = (X_{uw})_v\]  
\[\text{"}\]

\[(N_{uv})_u = (N_{uw})_v\]  
\[\text{"}\]

First one:

\[\left( \Gamma^1_{11} X_{uv} + \Gamma^2_{11} Y_{uv} + e N \right)_v = \left( \Gamma^1_{12} X_{uv} + \Gamma^2_{12} Y_{uv} + f N \right)_u\]

\[= X_u \left( \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{12} + \Gamma^2_{11} \Gamma^2_{12} + e a_{12} \right) + X_v \left( \Gamma^2_{12} + \Gamma^1_{11} \Gamma^2_{12} + \Gamma^2_{11} \Gamma^2_{22} + e a_{22} \right) + N \left( e v + \Gamma^1_{11} f + \Gamma^2_{11} g + 0 \right) + N \left( f v + \Gamma^1_{12} e + \Gamma^2_{12} f \right)\]

\[X_v \text{ coeff.} = e a_{22} - f a_{21} = \Gamma^2_{12,u} - \Gamma^2_{12,v} + P(\Gamma^k_{ij}) \text{, degree 2}\]

\[\text{LHS} = \frac{1}{E G - F^2} \left( e (F F - g E) - f (g F - f E) \right)\]

\[= -E \cdot \frac{e g - f^2}{E G - F^2}\]

Thus \[K = \frac{e g - f^2}{E G - F^2} = \frac{1}{E} \left( \Gamma^2_{12,u} - \Gamma^2_{12,v} - P(\Gamma^k_{ij}) \right)\]

a function of \(E, F, G\) and their derivatives, although the definition of \(K\) uses \(e, f, g\).

Theorema egregium

Theorem (Gauss): The Gaussian curvature \(K\) of a surface is invariant by local isometries.
Mainardi-Codazzi eq:

\[
\begin{align*}
&ev - fu = e \Gamma_{12} + f(\Gamma_{12} - \Gamma_{11}) - \Gamma_{11} g \\
&fv - gu = e \Gamma_{22} + f(\Gamma_{22} - \Gamma_{21}) - \Gamma_{21} g
\end{align*}
\]

Do we have other compatibility conditions? No

**Theorem**

If $E, F, G, e, f, g$ are smooth functions on $V \subset \mathbb{R}^2$ satisfying $E > 0, G > 0, EG - F^2 > 0$, the Gauss and M-C equations. Then for every $v \in V$, $e \cup e \in V$ and $X: U \to x(u) \subset \mathbb{R}^3$ with given coefficients for 1st and 2nd fundamental forms. It is unique up to translation, reflection, and rotation.

\[ R_k \]

Conditions from $(Nv)_u = (Ny)_v$ can be derived from Gauss formulas & Codazzi eq.
§4.4. Parallel transport and geodesics

(tangential)

Let \( \mathbf{w}(p) \in T_pS \) be a vector field defined in \( p \in U \) open \( c \ S \).

Let \( \alpha: (-\varepsilon, \varepsilon) \to S \) be a parametrized curve on \( S \).

**Definition:** The **covariant derivative** at \( p = \alpha(0) \) of \( \mathbf{w} \) along direction \( \mathbf{y} = \alpha'(0) \) is:

\[
(D_\mathbf{y} \mathbf{w})(p) = \frac{d\mathbf{w}}{dt}(0) = T_p \frac{d\mathbf{w}}{dt}(0),
\]

where \( \mathbf{w}(t) = \mathbf{w}(\alpha(t)) \), and

where \( T_p \) is the normal projection onto \( T_p(S) \), with normal \( \mathbf{N} \)

\[
T_p \mathbf{v} = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{N}}{\mathbf{N} \cdot \mathbf{N}} \mathbf{N}
\]

If \( S \subset \mathbb{R}^2 \), \( \frac{d\mathbf{w}}{dt} = \frac{d\mathbf{w}}{dt} \).

If \( \alpha(t) = X(u(t), v(t)) \)

\( \mathbf{w}(t) = \alpha(u(t), v(t)) \mathbf{X}_u + b(u(t), v(t)) \mathbf{X}_v = \alpha(t) \mathbf{X}_u + b(t) \mathbf{X}_v \)

then

\[
\frac{d\mathbf{w}}{dt} = \alpha (X_{uu} \mathbf{u}' + X_{uv} \mathbf{v}') + b (X_{vu} \mathbf{u}' + X_{vv} \mathbf{v}')
\]

\[
+ a' \mathbf{X}_u + b' \mathbf{X}_v
\]

\[
X_{uu} = \Gamma_{11}^1 \mathbf{X}_u + \Gamma_{11}^2 \mathbf{X}_v + \varepsilon \mathbf{N} \quad \text{etc}
\]

\[
\frac{d\mathbf{w}}{dt} = \mathbf{X}_u \left( \alpha(\Gamma_{11}^1 \mathbf{u}' + \Gamma_{12}^1 \mathbf{v}') + b(\Gamma_{12}^1 \mathbf{u}' + \Gamma_{22}^1 \mathbf{v}') + a' \right)
\]

\[
+ \mathbf{X}_v \left( \alpha(\Gamma_{11}^2 \mathbf{u}' + \Gamma_{12}^2 \mathbf{v}') + b(\Gamma_{12}^2 \mathbf{u}' + \Gamma_{22}^2 \mathbf{v}') + b' \right)
\]
1. The notation $D_y W(p)$ avoids $\alpha(t)$.

   Dependence on $\alpha(t)$ is only through $y = (u', v')$.
   If $\beta : (-\infty, \infty) \to S$, $\beta(0) = p$, $\beta'(0) = y$, same $D_y W(p)$.

2. It only depends on $\Gamma^k_{ij}$, i.e. 1st fund. form.

   Thus it is intrinsic, indep. of how $S$ is "immersed" in $\mathbb{R}^3$.

   Covariant derivative

3. The above can be extended to $W(t)$ only defined along $\alpha(t) = x(u(t), v(t))$,

\[ W(t) = \alpha(t) X_u + b(t) X_v. \]

   It may be undefined outside the curve.
   In this case, $\frac{D W}{d t}$ is a better notation.

Example: $W(t) = \alpha'(t)$.

\[ \frac{D \alpha'}{d t} = \Pi \alpha'' \]

   tangential component of acceleration $\alpha''$ of a particle
   moving along $\alpha(t)$.
**Defn** A vector field \( \mathbf{w} \) along a parametrized curve \( \alpha: \mathbb{I} \to S \) is **parallel** if \( \frac{D \mathbf{w}}{dt} = 0 \) \( \forall t \in \mathbb{I} \).

**Rk.** Externally, \( \frac{D \mathbf{w}}{dt} = 0 \iff \frac{d \mathbf{w}}{dt} = c(t) \mathbf{N} \).

**Ex.1** Consider a parallel of colatitude \( \phi \) on the unit sphere.

\[
\alpha(t) = (\sin \phi \cos t, \sin \phi \sin t, \cos \phi)
\]

\[
\alpha'(t) = (-\sin \phi \sin t, \sin \phi \cos t, 0)
\]

\[
\alpha''(t) = (-\sin \phi \cos t, -\sin \phi \sin t, 0)
\]

\[
N(t) = \alpha(t), \quad |N(t)| = 1
\]

\[
\frac{D \alpha'(t)}{dt} = \alpha'' - (\alpha'' \cdot N) N = \alpha'' + \sin^2 \phi \alpha
\]

\( \alpha' \) is parallel along \( \alpha \)

\( \iff \frac{D \alpha'}{dt} = 0 \iff \cos \phi = 0 \) i.e. \( \phi = \frac{\pi}{2} \).

**Rk.** In Ex.1, \( |\alpha'| = 1 \), thus \( \alpha'' \cdot \alpha' = 0 \)

\( \Rightarrow \frac{D \alpha'}{dt} \parallel \alpha', N, \quad \frac{D \alpha'}{dt} \parallel X \phi \)

In fact, \( \frac{D \alpha'}{dt} = -\sin \phi \cos \phi X \phi \).
Prop.1 If \( W \) \& \( V \) are parallel vector fields along \( \alpha : I \to S \), then \( \langle W(t), V(t) \rangle \) is constant.

In particular, \( |W(t)| = \text{const.} \), angle between \( W(t) \& V(t) \) is const.

**Proof.** \[ W'(t) = c(t) N(t), \quad V'(t) = d(t) N(t) \]

\[
\frac{d}{dt} \langle W(t), V(t) \rangle = \langle W', V \rangle + \langle W, V' \rangle
\]

\[ = \langle cN, V \rangle + \langle W, dN \rangle = 0. \quad \Box \]

**Rk.** Intrinsically, for vector fields not necessarily parallel,

\[
\frac{d}{dt} \langle W(t), V(t) \rangle = \langle \frac{dW}{dt}, V \rangle + \langle W, \frac{dV}{dt} \rangle
\]

This can provide an intrinsic proof.

Prop.2 Let \( \alpha : I \to S \), \( t_0 \in I \), \( p = \alpha(t_0), \ W_0 \in T_p(S) \).

Then there exists a unique parallel vector field \( W(t) \) along \( \alpha(t) \), with \( W(t_0) = W_0 \).

The vector \( W(t_1), t_1 \in I \), is the parallel transport of \( W_0 \) along \( \alpha \) at \( t_1 \).

**Pf.** Unique existence of ODE \( a' = \cdots, b' = \cdots \) from \( \alpha(0) = 0 \).
1. Independent of parametrization. If \( x(t) = \beta(\sigma), \)
\( t = t(\sigma), \) then
\[
\frac{DW}{d\sigma} = \frac{DW}{dt} \cdot \frac{dt}{d\sigma}
\]
They vanish together.

2. Parallel transport: for \( t_0, t_1 \in I, \)
\[
P_{t_0} : T_{x(t_0)} S \rightarrow T_{x(t_1)} S
\]
is an isometry by Prop 1.

3. If two surfaces \( S, \bar{S} \) are tangent along a curve \( x : I \rightarrow S \cap \bar{S}, \)
so that they share same \( N(t), \) then
- \( \frac{DW}{dt} \) is the same for both, \( \left( \frac{DW}{dt} = \frac{W'}{N \cdot N} \right) \)
- parallel transport is the same for both.

Ex2: we continue Ex1

At \( x(0) = (\sin \phi, 0, \cos \phi), \) choose \( W_0 = x'(0) = (0, \sin \phi, 0) \)
In spherical coordinates \((\phi, \theta),\)
Suppose its parallel transport is

\[
W(t) = a(t) x_0 + b(t) x_\phi, \quad x_0 = \sin \phi
\]
\( a(0) = 1, \ b(0) = 0. \)
\[
x_\theta = x(t), \quad \frac{DX_0}{dt} = x'' + \sin^2 \phi \dot{x} = -\sin \phi \cos \phi \quad x_\phi
\]
\( x_\phi = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi), \quad |x_\phi| = 1 \)
\[
\frac{dx_\phi}{dt} = (\cos \phi \sin \theta, \cos \phi \cos \theta, 0) = \cot \phi \quad \alpha'
\]
\[
\frac{DX_\phi}{dt} = \cot \phi \quad x_0
\]