Curvature of curves on a surface

**Definition**

Let $C$ be a regular curve in a regular surface $S$. At $p \in C$, let $N$ be normal of $S$

$n \parallel C$  \hspace{1cm}  $k$ curvature of $C$  

$\gamma$ ... angle between $N$ & $n$  

$\cos \gamma = N \cdot n$  

The normal curvature of $C \subset S$ at $p$ is  

$$k_N = k \cdot \cos \gamma = kn \cdot N$$

length of $\text{proj}_N kn$

---

**Example**

Sphere $x^2 + y^2 + z^2 = a^2$

At $p = (x, y, z)$, with polar coordinates $(a, \phi, \theta)$

$$N = \frac{1}{a}(x, y, z)$$

$$r = \frac{\sqrt{x^2 + y^2}}{a} = \sin \theta$$

$$k = \frac{1}{r}$$

$$k_N = \frac{r^2}{a} = \frac{1}{a} (x, y, z) = \frac{1}{a}$$

$$\cos \gamma = N \cdot n = \frac{-r^2}{a} = -\frac{1}{a} = -\sin \theta$$

$$\delta = \theta + \frac{\pi}{2}$$

---

**RK1** If we had taken inward $N = -\frac{1}{a}(x, y, z)$, then $\delta = \frac{\pi}{2} - \theta$.

2. If a plane cut the sphere at a circle of radius $r$, then $k = \frac{1}{r}$, $k_N = \frac{1}{a}$ even if the plane is not horizontal.
Let us parameterize \( C \) by \( \alpha(s) \) where \( s \) is arclength of \( C \), \( \alpha(0) = p \).

Let \( N(s) = N(\alpha(s)) \). Note \( \alpha'(0) \in T_p(S) \), \( |\alpha'(s)| = 1 \).

\[
\Pi_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = -\langle N'(0), \alpha'(0) \rangle = 0
\]

\[
\left. \frac{d}{ds} \right|_{s=0} \langle N, \alpha' \rangle + \langle N(0), \alpha'(0) \rangle = 0
\]

In particular, \( k_n \) only depends on \( \alpha'(0) \), unlike \( k \). This explains the choice of sign of \( \Pi_p \).

Prop 2 (Meusnier) The normal curvature of any curve \( \alpha(s) \) in \( S \) at \( p = \alpha(0) \) with \( \alpha' \) an arclength is

\[
k_n = \Pi_p(\alpha'(0)).
\]

It agrees with any other curve \( \beta(s) \) parametrized by arclength \( \beta(0) = p \) and \( \beta'(0) = \alpha'(0) \).

We may take \( \beta(s) \) the normal section of \( S \) at \( p \) along \( \nu = \alpha'(0) \), i.e., \( \beta(s) \) is the intersection of \( S \) with the plane passing \( p \) spanned by \( N \) & \( \nu \). Then \( N \perp N \), \( k_n = \pm k \), the normal curvature only depends on unit vectors in \( T_p(S) \).
Ex 1. on a plane, all normal sections are straight lines, all normal curvature = 0.

Ex 2. on sphere $S_a: x^2 + y^2 + z^2 = a^2$, all normal sections are great circles, i.e., circles on $S_a$ with radius $a$ and same centre.

$K_n = \frac{1}{a}$ along any $v \in T_p(S_a)$

$\Pi_p(v) = 1$ if $|v| = 1$

Ex 3. cylinder $S: x^2 + y^2 = a^2$

$N(x, y, z) = -\frac{1}{a}(x, y, 0)$

The plane spanned by $N, v$ cut $S$ at an ellipse, except when $v = (0, 0, v_3)$,

$v \perp e_3$: horizontal circle; $K_n = \frac{1}{a}$

$v \parallel e_3$: vertical line; $K_n = 0$

Recall $dN_p$, e-vector $e_3$, e-value 0, e-vector $e_\theta = \left(\frac{-v}{a}, \frac{x}{a}, 0\right)$, e-value $-\frac{1}{a}$

$\Pi_p(v) = \langle -dN_p(v), v \rangle = \cos^2 \frac{1}{a} + \sin^2 \frac{1}{a} \in [0, \frac{1}{a}]$.
\[ z = \frac{a}{2} x^2 + \frac{b}{2} y^2, \quad a > b, \quad \text{may be } \frac{a}{2} 0 < 0 \]

At \( p = (0, 0, 0) \),
\[ dN_p(e_1) = -ae_1, \quad dN_p(e_2) = -be_2, \quad e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0) \]

If \( \mathbf{v} \in T_p(S) \), \( |\mathbf{v}| = 1 \) \implies \( \mathbf{v} = \cos \alpha e_1 + \sin \alpha e_2 \) some \( \alpha \)

\[ \Pi_p(\mathbf{v}) = \langle -dN_p(\mathbf{v}), \mathbf{v} \rangle = \cos^2 \alpha a + \sin^2 \alpha b \]
\[ \in [b, a] \]

In general, if at \( p \in S \),
\[ dN_p(e_1) = -k_1 e_1, \quad dN_p(e_2) = -k_2 e_2 \]
for some orthonormal basis \( \{e_1, e_2\} \) of \( T_p(S) \), \( k_1 > k_2 \), then

\[ k_1 = \max_{|\mathbf{v}| = 1} \Pi_p(\mathbf{v}) = \max \text{ normal curvature} \]

\[ k_2 = \min \text{ normal curvature} \]

\textbf{Defn} \quad k_1 \text{ and } k_2 \text{ above are principle curvatures at } p.
\text{ \( e_1 \) and } e_2 \text{ above are principle directions at } p.

\textbf{Umbilical}

\text{If } k_1 = k_2 \text{ (e.g., plane & sphere) choice of } e_1, e_2 \text{ nonunique.}

\textbf{Euler formula:}

\text{Normal curvature } k_n \text{ along } V = e_1 \cos \alpha + e_2 \sin \alpha \text{ is}

\[ k_n = \Pi_p(V) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha \]
Defn A regular connected curve $C$ on $S$ is a line of curvature if the tangent at any point $p$ of $C$ is a principle direction at $p$.

In this case, for any parametrization $\alpha(t)$ of $C$,

\[
d'\alpha(t) \perp dN(t), \quad \text{then} \quad dN'\alpha(t) = \kappa(t) d\alpha(t) \quad \text{for some $\kappa(t)$}.
\]

Conversely, if $\kappa(t)$ is true for one regular parametrization, then $C$ is a line of curvature.

\[ H = \frac{k_1 + k_2}{2} \]

Mean curvature = average

Note: The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of $-dN_p p$ relative to any basis $\{v_1, v_2\}$ of $T_p S$ depends on choice of $B$.

But its eigenvalues $k_1, k_2$, and hence $K$ of $H$, do not.

In fact, Moreover, when we switch orientation $N \rightarrow \overline{N}$, $k_1, k_2$ and $H$ change sign, but $K$ does not.

Defn A pt $p$ of a surface $S$ is

- elliptic, if $K > 0$ ($k_1, k_2$ same sign)
- hyperbolic if $K < 0$ ($k_1 > 0, k_2 < 0$)
- parabolic if $K = 0$, $dN_p \neq 0$ (exactly one of $k_1, k_2 = 0$)
- planar if $dN_p = 0$ ($k_1 = k_2 = 0$)
- umbilical if $k_1 = k_2$
Examples:

- \( z = x^2 + y^2 \), \( z = -x^2 - y^2 \)
- \( z = x^2 - y^2 \)
- \( z = x \), \( z = x^4 + y^4 \) at \( \partial \)
- \( z = \frac{k}{2} (x^2 + y^2) \), \( k = 0 \) or \( k \neq 0 \).
- \( x^2 + y^2 + z^2 = a^2 \)

**Prop.** If all points of a connected surface \( S \) are umbilical pt., then \( S \) is contained in a plane or a sphere.

**proof.** \( \forall q \), \( \exists \lambda(q) \) s.t.

\[
dN_q(w) = \lambda(q) \cdot w \quad \forall w \in T_q(S)
\]

**claim:** \( \lambda(q) = \text{const.} \). For any parametrization \( \vec{x}(u,v) : \mathbb{R}^2 \to S \), let \( N(u,v) = N(\vec{x}(u,v)) \).

\[
\lambda(q) \vec{x}_u = dN_q(\vec{x}_u) = N_u,
\]

\[
N_u = \lambda(q) \vec{x}_u, \quad N_v = \lambda(q) \vec{x}_v
\]

\[
(\lambda(q) \vec{x}_u) \cdot v = N_u \cdot v = N_v \cdot u = (\lambda(q) \vec{x}_v) \cdot u
\]

\[
\lambda_v \vec{x}_u = \lambda_u \vec{x}_v
\]

\( \Rightarrow \) \( \lambda_u \vec{x}_u = 0 \), \( \lambda = \text{const.} \) or trace of \( \vec{x} \).

Since \( \vec{x} \) is arbitrary, \( \lambda = \text{const.} \) on \( S \).

If \( \lambda = 0 \), \( \lambda = 0 \), \( N = N_o \) in trace of \( \vec{x} \), hence in \( S \).

\( \Rightarrow \) \( S \) is a plane.

If \( \lambda \neq 0 \),

\[
(\vec{x} - \frac{1}{\lambda} N)\cdot u = 0, \quad (\vec{x} - \frac{1}{\lambda} N)\cdot v = 0
\]

\( \Rightarrow \) \( \vec{x} - \frac{1}{\lambda} N \equiv \vec{x}_o \) on \( V \)