A parametrization of the Möbius strip.

\[
X(t, \theta) = \left( (1 - t \sin \frac{\theta}{2}) \cos \theta, (1 - t \sin \frac{\theta}{2}) \sin \theta, t \cos \frac{\theta}{2} \right) \quad \text{--- regular surface}
\]

\[-\frac{1}{2} < t < \frac{1}{2}, \quad 0 \leq \theta \leq 2\pi\]

\[X(0, \theta) = (\cos \theta, \sin \theta, 0), \quad \text{Unit circle on xy-plane}\]

**Def.** Given an orientation \( N \) on a regular surface \( S \), a basis \( \{ U, W \} \subset T_p(S) \) is **positive** (positively oriented) if \( U \times W \) is in same direction of \( N \), \( \langle U \times W, N \rangle > 0 \).

**Def.** Let \( S \) be a regular surface with orientation \( N \). Its **Gauss map** is

\[N : S \rightarrow S^2,\]

* A map between surfaces
* The Gauss map is smooth (since \( N : S \rightarrow \mathbb{R}^3 \) was)
* Its differential \( dN_p : T_p(S) \rightarrow T_{N(p)}(S^2) \) is identified

A tangent vector \( U(t) \) is mapped to another tangent vector \( \frac{d}{dt} \bigg|_{t=0} N(U(t)) \)

In particular, it gives info on how the surface is curved.
**Ex1** A plane, \( N = \text{const} \), \( \text{d}N = 0 \), not curved.

**Ex2** Sphere of radius \( a \)

\[
S = x^2 + y^2 + z^2 = a^2
\]

Inward normal \( N(x, y, z) = \left( -\frac{x}{a}, -\frac{y}{a}, -\frac{z}{a} \right) \)

It agrees with unit normal vector of any great circle on \( S \).

If \( \alpha(t) = (x(t), y(t), z(t)) \) is a curve on \( S \),

\[
\frac{dN(\alpha(t))}{dt} N(\alpha(t)) = -\frac{1}{a} \alpha'(t)
\]

\( dN = \alpha'(0) \rightarrow -\frac{1}{a} \alpha'(0) \), \( dN = -\frac{1}{a} \text{Id} \) on \( T_p(S) \)

If \( a \gg 1 \), \( \frac{1}{a} \ll 1 \), slightly curved.

If \( a \ll 1 \), \( \frac{1}{a} \gg 1 \), very curved.

If we had chosen outward \( \bar{N}(x, y, z) = \left( \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right) \), \( \bar{d}N = \frac{1}{a} \text{Id} \).

**Ex3** Cylinder \( S \); \( x^2 + y^2 = a^2 \)

\[
N(x, y, z) = \left( -\frac{x}{a}, -\frac{y}{a}, 0 \right)
\]

\( \alpha(t) = (x(t), y(t), z(t)) \), \( \rho = \alpha(0) = (x_0, y_0, z_0) \)

\[
\frac{dN(\alpha(t))}{dt} N(\alpha(t)) = \left( -\frac{x'}{a}, -\frac{y'}{a}, 0 \right)
\]

Consider \( v, w \in T_p(S) \)

- IF \( v = c(0, 0, 1) \), \( \rho N(v) = 0 \), \( v \ldots \text{eigen vector with e-value } 0 \)
- IF \( w = c(-\frac{y_0}{a}, \frac{x_0}{a}, 0) \), \( w \cdot N = 0 \)

\[
\frac{dN(w)}{dt} = -\frac{1}{a} w \quad \text{w...eigen vector with e-value } -\frac{1}{a}.
\]
Ex. 4  \( S: Z = \frac{ax^2 + by^2}{2} = f(x, y) \)

If \( a > 0, b > 0 \) ---- paraboloid.

\( a > 0 > b \) ---- hyperboloid.

Parametrization \( \vec{X}(u, v) = (u, v, f(u, v)) \)

\( \vec{X}_u = (1, 0, au) \),
\( \vec{X}_v = (0, 1, bv) \)

\( \vec{X}_u \times \vec{X}_v = (-au, -bv, 1) \)

\( N(u, v) = \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|} = \frac{(-au, -bv, 1)}{\sqrt{a^2u^2 + b^2v^2 + 1}} \)

At \( P \), \( T_P(S) \) is horizontal.

If \( \alpha(t) = \vec{X}(u(t), v(t)) \)

(i). \( u(t) = t, v(t) = 0, \alpha'(0) = X_u = (1, 0, 0) = e_1 \)

\( dN_P(e_1) = \frac{d}{dt}\bigg|_{t=0} N(\alpha(t)) = \frac{d}{dt}\bigg|_{t=0} \frac{(-at, 0, 1)}{\sqrt{a^2t^2 + 1}} = \frac{(-a, 0, 0)}{1} + \frac{(at, 0, 1)}{2(a^2t^2 + 1)^{3/2}} (-2at) \)

\( = -ae_1, \quad e_1 \) is an eigenvector with e-value \( -a \).

(ii). \( u(t) = 0, v(t) = 1, \alpha'(0) = X_v = (0, 1, 0) = e_2 \)

\( dN_P(e_2) = \frac{d}{dt}\bigg|_{t=0} N(\alpha(t)) = \frac{d}{dt}\bigg|_{t=0} \frac{(0, -bt, 1)}{\sqrt{b^2t^2 + 1}} = -be_2 \) by similar computation.

\( e_2 \) is an eigenvector with e-value \( -b \).
Prop 1 Let $S$ be a regular surface with unit normal vector field $N$ normal to $S$. The differential $dN_p: T_p(S) \rightarrow T_p(S)$ of the Gauss map is a self-adjoint map, i.e.,

$$\langle v, dN_p(w) \rangle = \langle dN_p(v), w \rangle \quad \forall \, v, w \in T_p(S)$$

Proof. Let $\vec{x}(u,v)$ be a parametrization of $S$ at $p = \vec{x}(0,0)$. Then $\{\vec{x}_u, \vec{x}_v\}$ is a basis of $T_p(S)$.

Since $dN_p$ is linear, it suffices to show

$$\langle \vec{x}_u, dN_p(\vec{x}_v) \rangle_p = \langle dN(\vec{x}_u), \vec{x}_v \rangle_p$$

$$\vec{x}_u = \frac{d}{dt} \big|_{t=0} \vec{x}(t,0)$$

$$dN(\vec{x}_u) = \frac{d}{dt} \bigg|_{t=0} N(\vec{x}(t,0)) = N_u, \quad \text{for} \quad N(u,v) = N(\vec{x}(u,v))$$

$$dN_p(\vec{x}_v) = N_v \quad \text{similarly}.$$}

$$\langle \vec{x}_u, N_v \rangle = \langle N_u, \vec{x}_v \rangle = ?$$

LHS = $\partial_v \langle \vec{x}_u, N \rangle - \langle \vec{x}_u, \partial_u N \rangle = 0 - \langle \vec{x}_u, \partial_v N \rangle$

RHS = $\partial_u \langle N, \vec{x}_v \rangle - \langle \partial_u N, \vec{x}_v \rangle = 0 - \langle \partial_u N, \vec{x}_v \rangle$

Corollary 1 The bilinear form

$$B(v,w) = \langle dN_p(v), w \rangle, \quad v, w \in T_p(S)$$

is thus symmetric, $B(v,w) = B(w,v)$.

(It is bilinear in the sense that it is linear in both $v$ and $w$.)
Another consequence

If we choose an orthonormal basis \( \{e_1, e_2\} \) for \( T_p(S) \), and let 
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]
be the matrix relative to this basis,

\[
dN(e_1) = a_{11} e_1 + a_{12} e_2 = (e_1, e_2) \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \quad (1)
\]
\[
dN(e_2) = a_{21} e_1 + a_{22} e_2 = (e_1, e_2) \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \quad (2)
\]
\[
dN(c_1 e_1 + c_2 e_2) = (e_1, e_2) A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (3)
\]

By Prop 1, 
\[
a_{12} = \langle dN(e_1), e_2 \rangle = \langle e_1, dN(e_2) \rangle = a_{21}
\]

Thus \( A \) is symmetric.

By linear algebra, \( A \) is orthogonally diagonalizable, i.e.,
(Direct proof for \( \dim 2 \) in textbook p.218)

there are eigen vectors \( V_1 = (v_{11}, v_{12}) \) & \( V_2 = (v_{21}, v_{22}) \) of \( A \) s.t.

\[
A V_1 = \lambda_1 V_1, \quad A V_2 = \lambda_2 V_2,
\]

\[
|V_{11}| = |V_{21}| = 1, \quad V_1 \cdot V_2 = 0
\]

Let \( \hat{e}_1 = V_{11} e_1 + V_{12} e_2 \), \( \hat{e}_2 = V_{21} e_1 + V_{22} e_2 \in T_p(S) \)

\[
dN(\hat{e}_1) = (e_1, e_1) \cdot A \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = (e_1, e_1) \lambda_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \lambda_1 \hat{e}_1
\]

Similarly \( dN(\hat{e}_2) = \lambda_2 \hat{e}_2 \)

\[
|\hat{e}_1| = |\hat{e}_2| = 1, \quad \langle \hat{e}_1, \hat{e}_2 \rangle = 0
\]

We may skip the hat.

**Corollary 2** There are orthonormal basis \( e_1, e_2 \in T_p(S) \)

consisting of eigenvectors of \( dN_p \).
**Definition 2**  The second fundamental form of $S$ at $p$ is

$$\Pi_p(v) = -\langle dN_p(v), v \rangle, \quad v \in T_p(S).$$

**RK2**  If $v = x e_1 + y e_2$, with $dN(e_1) = \lambda_1 e_1$, $dN(e_2) = \lambda_2 e_2$, then

$$\Pi_p(v) = -\langle x \lambda_1 e_1 + y \lambda_2 e_2, x e_1 + y e_2 \rangle$$

$$= -\lambda_1 x^2 - \lambda_2 y^2$$

**RK1**  we can recover the bilinear form $B(v, w)$:

$$\Pi_p(v+w) = -B(v+w, v+w) = \Pi_p(v) - 2B(v, w) + \Pi_p(w)$$

Hence $\Pi_p$ carries as much info as $B$. 
