We talked about the differential $dx$ of $\mathbb{R}^n \to \mathbb{R}^m$.

We now consider the differential of a map between surfaces.

Let $S_1$ and $S_2$ be regular surfaces and $\varphi : V \subset S_1 \to S_2$ be a smooth map. For $p \in V$ and any $v \in T_p(S_1)$, we choose

$$d : (-\varepsilon, \varepsilon) \to S_1, \quad \alpha(0) = p, \quad \alpha'(0) = v$$

and let $\beta(t) = \varphi \circ \alpha(t)$. Then $\beta'(0)$ is a tangent vector at $q = \varphi(p)$. We define the map

$$d \varphi_p : T_p(S_1) \to T_q(S_2),
\quad v \mapsto \beta'(0),$$

**Prop 2** For given $w$, the vector $\beta'(0)$ does not depend on the choice of $d(t)$. The map $d \varphi_p$ is linear. It is called the differential of $\varphi : S_1 \to S_2$ at $p$. 
Proof of Prop 2

Let \( \varphi: U_1 \subset \mathbb{R}^2 \to S_1 \) be parametrizations of \( S_1 \) of \( S_2 \) with \( p \in \varphi(U_1) \) and \( q = \varphi(p) \in \varphi(U_2) \).

We may take \( U_1 \) sufficiently small so that \( \varphi \circ \chi(U_1) \subset \varphi(U_2) \).

Let

\[ \Psi = \varphi^{-1} \circ \varphi \circ \chi : U_1 \to U_2 \]

\( \Psi \) is smooth by the definition of \( \varphi \) being a smooth map. \( \Psi \)'s being a smooth map.

For any curve

\[ \alpha : (-\varepsilon, \varepsilon) \to S_1, \quad \alpha(0) = p, \quad \alpha'(0) = w \]

let

\[ \beta = \Psi \circ \alpha = \varphi \circ \varphi^{-1} \circ \chi \circ \alpha \]

Thus

\[ \beta'(0) = \frac{d}{dt}_{t=0} \Psi \circ \alpha(t) = dy_{S_1} \cdot dy_{S_2} \cdot dx^1(w) \]

is uniquely defined, and is linear in \( w \).
Ex (Rotation of the sphere)

Let \( R_\theta : \mathbb{R}^3 \to \mathbb{R}^3 \) be the rotation of angle \( \theta \) about the z-axis.

\[
R_\theta(x^T) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} x^T, \quad x \in \mathbb{R}^3
\]

Consider its restriction to the unit sphere \( S^2 = \{ x \in \mathbb{R}^3, |x|^2 = 1 \} \).

Being the restriction of a smooth map,

\( R_\theta : S^2 \to S^2 \)

is a smooth map. We want to compute \( (dR_\theta)_p(w) \) fn \( p \in S^2, \quad w \in T_p(S^2) \).

Choose \( \alpha : (-\varepsilon, \varepsilon) \to S^2 \),

\( \alpha(0) = p, \quad \alpha'(0) = w \)

Let \( \beta(t) = R_\theta(\alpha(t)) \)

\( (dR_\theta)_p(w) = \beta'(0) = R_\theta \alpha'(0) = R_\theta w \)

\( \Rightarrow (dR_\theta)_p = R_\theta \)

\( \text{Rk} \quad 1. \text{This is true for any linear map } T : \mathbb{R}^3 \to \mathbb{R}^3 \)

2. If \( p = N = (0,0,1) \) north pole,

\( R_\theta p = p \)

\( T_p(S^2) = \{ (x,y,z) : x,y \in \mathbb{R}, z = 1 \} \)

\( (dR_\theta)_N \) is a rotation on \( \mathbb{R}^2 \)
Consider the nonlinear map \(\Phi\) (double cover)

\[ \Phi : S \rightarrow S \]

\[ \Phi (\cos t, \sin t, z) = (\cos 2t, \sin 2t, z) \]

\(\Phi\) is well-defined: \(\Phi(\cos(t+2\pi), \sin(t+2\pi), z) = \Phi(\cos t, \sin t, z)\)

\(\Phi\) is continuous:

\[ \Phi(\cos 0, \sin 0, z) = (1, 0, z) \]

\[ \Phi(\cos 2\pi, \sin 2\pi, z) = (1, 0, z) \]

It covers \(S\) twice.

\[ \Phi(\cos(t+2\pi), \sin(t+2\pi), z) = \Phi(\cos t, \sin t, z) \]

Q: \(d\Phi|_p = ?\)

\[ e_z = (0, 0, 1), \quad e_\theta = (0, 1, 0) \]

If \(p = (1, 0, z_0)\), \(\Phi(p) = p\), \(d\Phi|_p(e_z) = e_z\), \(d\Phi|_p(e_\theta) = 2e_\theta\)

\(\alpha(t) = (1, 0, z_0 + t)\) and \(\beta(t) = (\cos t, \sin t, z_0)\); \(\gamma(t) = (1, 0, z_0 + t)\) and \(\delta(t) = (\cos t, \sin t, z_0)\)

For general \(p = (\cos \theta, \sin \theta, z)\), \(0 \leq \theta < 2\pi\)

\[ e_z = (0, 0, 1) \]

\[ e_\theta = (-\sin \theta, \cos \theta, 0) \]

\[ T_p(S) = \text{span}\{e_\theta, e_z\} \]

\[ d\Phi|_p : \begin{cases} e_z \rightarrow e_z, \\ e_\theta \rightarrow 2e_\theta \end{cases} \]

\[ a e_\theta + b e_z \rightarrow 2a e_\theta + b e_z \]
Normal vector:

Given a pt \( p \) on a regular surface \( S \),

there are 2 unit vectors in \( \mathbb{R}^3 \) that are orthogonal to \( T_p(S) \).

They are the unit normal vectors at \( p \). The line

\[ \alpha(t) = p + tN, \quad t \in \mathbb{R} \]

is the normal line at \( p \).

For any parametrization

\[ \mathbf{x}: U \subset \mathbb{R}^2 \rightarrow S, \]

\[ q = \mathbf{x}(u, v) \]

A unit normal vector at \( q \) is

\[ N(q) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}(q) \]

\( |\mathbf{x}_u \times \mathbf{x}_v| \) is smooth as it stays away from 0.

This defines a smooth normal vector field \( N(q) \) on \( \mathbf{x}(U) \subset S \). This may be impossible on entire \( S \), such as on a Möbius band.
Chapter 2.5 First fundamental form

It enables us to compute length, angle & area on surfaces.

**Defn** Let \( S \subset \mathbb{R}^3 \) be a regular surface. The first fundamental form \( I_p \) of \( S \) at \( p \in S \) is

\[
I_p : T_p(S) \to \mathbb{IR}
\]

\[
I_p (w) = \langle w, w \rangle_p = |w|^2 .
\]

Here \( \langle , \rangle_p \) is the induced inner product on \( T_p(S) \) from that of \( \mathbb{R}^3 \):

\[
\langle w, v \rangle_p = w \cdot v
\]

In the future, we talk about surfaces not in \( \mathbb{R}^3 \) and we still want \( I_p \) and \( \langle , \rangle_p \).

**Remark**

1. \( \langle v, w \rangle_p \) is bilinear. (Linear in both \( v \) & \( w \)), \( I_p(aw) = a^2 I_p(w) \)
2. \( I_p(w+v) = \langle w+v, w+v \rangle_p \)

\[
= \langle w, w \rangle_p + \langle w, v \rangle_p + \langle v, w \rangle_p + \langle v, v \rangle_p
\]

\[
= I_p(w) + I_p(v)
\]

We defined \( I_p \) by \( \langle , \rangle \). But once we have \( I_p \), we can define \( \langle , \rangle \) by

\[
\langle w, v \rangle_p = \frac{1}{2} \left( I_p(w+v) - I_p(w) - I_p(v) \right)
\]

Thus \( I_p \) and \( \langle , \rangle_p \) give same info.
First fundamental form relative to a parametrization:

\[ \vec{X} : U \subset \mathbb{R}^2 \rightarrow S, \quad p \in \vec{X}(U). \]

A basis of \( T_p(S) \) is \( \vec{X}_u, \vec{X}_v \).

Suppose \( \alpha : (-\varepsilon, \varepsilon) \rightarrow S \), \( \alpha(0) = p, \alpha'(0) = w \)

and assume

\[ \alpha(t) = \vec{X}(u(t), v(t)). \]

\[ w = \alpha'(0) = xu_u(0) + xv_v(0). \]

\[ I_p(w) = \ ? \]

\[ = \langle x_u w, x_v v \rangle \cdot \text{(itself)} \]

\[ = \langle x_u, x_u \rangle_p w(0)^2 + 2 \langle x_u, x_v \rangle_p w(0)v(0) + \langle x_v, x_v \rangle_p v(0)^2 \]

\[ = E(u_0, v_0) + F(u_0, v_0) + G(u_0, v_0) \]

\( E, F, G \ldots \) coefficients of \( I_p \)

Smooth functions in \( U \).