Definition 1b: Let \( S_1 \) and \( S_2 \) be regular surfaces. A continuous map
\[
\varphi: \mathcal{V}_1 \subset S_1 \to S_2
\]
is differentiable at \( p \in \mathcal{V}_1 \) if the map
\[
\begin{array}{ccc}
\mathcal{U}_1 & \xrightarrow{\varphi} & \mathcal{U}_2 \\
\mathcal{X}_1 & \mathcal{X}_2
\end{array}
\]
is smooth at \( \mathcal{X}_i^{-1}(p) \) for some given parametrizations \( \mathcal{X}_i: \mathcal{U}_i \subset \mathbb{R}^2 \to S_i \), \( i = 1, 2 \). A map \( \varphi: S_1 \to S_2 \) is a diffeomorphism (of surfaces) if \( \varphi \) is smooth and has a smooth inverse \( \varphi^{-1} \).

Remark 1: The definition is independent of choice of parametrizations \( \mathcal{X}_1, \mathcal{X}_2 \), using Prop 1 for both \( \mathcal{X}_1, \mathcal{X}_2 \).

2. If \( S_1 \) and \( S_2 \) are diffeomorphic, they are indistinguishable in differential geometry. (Instead of diffeomorphisms, one may consider isometries, affine maps, etc.)

Example 2: If \( \mathcal{X}: \mathcal{U} \subset \mathbb{R}^2 \to S \) is a parametrization,
then \( \mathcal{X}^{-1}: \mathcal{X}(\mathcal{U}) \to \mathcal{U} \) is a smooth map since the map
\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\mathcal{X}} & \mathcal{X}(\mathcal{U}) & \xrightarrow{\mathcal{X}^{-1}} & \mathcal{U} & \xrightarrow{id} & \mathcal{U}
\end{array}
\]
is smooth. Since \( \text{id} \circ \mathcal{X}^{-1} \circ \mathcal{X} = \text{id} \); \( \mathcal{U} \to \mathcal{U} \) is smooth.

Thus \( \mathcal{X}: \mathcal{U} \to \mathcal{X}(\mathcal{U}) \) is a diffeomorphism, we often identify them, and write \( f(u,v) \) for both \( f: S \to \mathbb{R} \) and \( f \circ \mathcal{X}: \mathcal{U} \to \mathcal{X}(\mathcal{U}) \)
Every regular surface is locally diffeomorphic to a plane.

Indeed, if \( p \in \text{a regular surface } S \), choose a parametrization

\[
\tilde{x} : U \subset \mathbb{R}^2 \rightarrow V \text{nbhd of } p,
\]

\( p = \tilde{x}(u) = S \cap V \)

choose a small ball \( B = B(q, r) \subset U, \ q = \tilde{x}^1(u) \),

Let \( \phi : \mathbb{R}^2 \rightarrow B(q, r) \)

\[
\phi(u, v) = q + \frac{r(u, v)}{\sqrt{1 + u^2 + v^2}}
\]

Then

\[
\phi \circ \tilde{x}^1|_B : \mathbb{R}^2 \rightarrow B \rightarrow \tilde{x}(B) \ni p
\]

is a diffeomorphism between \( \mathbb{R}^2 \) and \( \tilde{x}(B) \).
Exercise 3. Let $S_1$ and $S_2$ be regular surfaces. Suppose $S_1 \subset V$ an open set in $\mathbb{R}^3$, $\varphi = V \rightarrow \mathbb{R}^3$ smooth, and $\varphi(S_1) \subset S_2$. Then the restriction $\varphi|_{S_1} : S_1 \rightarrow S_2$ is smooth.

\[ x_2^{-1} \circ \varphi|_{S_1} \circ x_1 \in C^\infty \]

\[ = x_2^{-1} \circ \varphi \circ x_1 \quad \text{yes}. \]

3a. (Reflection)
Let $\sigma(x, y, z) = (x, y, -z)$ reflection w.r.t. $xy$-plane.

\[ p + \sigma(p) = 2\pi(p) \]
\[ \sigma(p) = 2\pi(p) - p \]

If $S$ is symmetric relative to $xy$-plane, i.e., $\sigma(S) = S$,
Then the map
\[ \sigma : S \rightarrow S \]
is smooth, since it's the restriction of $\sigma : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

It remains true to reflections relative to any plane.
3b. (rotation) Let \( R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \) be rotation about \( z \)-axis by angle \( \theta \). If a regular surface \( S \) is invariant under it, \( R_\theta(S) = S \), then the map

\[
R_\theta : S \to S
\]

is smooth, since it is the restriction of \( R_\theta : \mathbb{R}^2 \to \mathbb{R}^3 \).

Possible \( S \):

- a surface of revolution, \( R_\theta S = S \) \( \forall \theta \).
- Discrete version:
  \[
  \Theta = \frac{2\pi}{k}, \quad k \in \mathbb{N},
  \]
  \[
  S = \bigcup_{j=0}^{k} S_j, \quad S_j = R_\theta S_{j-1}
  \]

Such \( S \) may be or may not be connected.

* A connected example: In cylindrical coordinates

\( 0 \leq \theta < 2\pi \), \( 1 < r < 2 \)

\[
x(\theta, r) = (r \cos \theta, r \sin \theta, \cos \frac{\theta}{k})
\]

For fixed \( r \), \( x(r, \theta) \) is a "periodic" (in \( \theta \)) curve surrounding \( z \)-axis. One then extend in \( r \) direction.

* A disconnected example: Water wheel
(2nd part of §2.3)
So far we have considered

- parametrized curve \( \mathbf{c}(t) : (a,b) \to \mathbb{R}^3 \)
- surface \( S \subset \mathbb{R}^3 \)

To do:
- curve
- parametrized surface

**Def** A subset \( C \subset \mathbb{R}^3 \) is a regular curve if \( \forall p \in C \), there is a nbhd \( V \) of \( p \) in \( \mathbb{R}^3 \) and a smooth, \( 1 \)-\( 1 \)

\[ \mathbf{c} : (a,b) \to V \]

such that \( \mathbf{c}((a,b)) = C \cap V \), \( \mathbf{c} \) has a continuous inverse on \( \mathbf{c}((a,b)) \), and \( \frac{d}{dt} \mathbf{c} \) has rank 1 \( \forall t \in (a,b) \).

- It is indep. of choice of parametrization \( \mathbf{c}(t) \), because any change of parameters is a diffeomorphism between 2 intervals.

- Local properties of \( C \): a property from parametrization but indep. of choice of parametrization, such as curvature and absolute value of torsion.

- Non-examples: orientation of \( C \), sign \( C \),

  global properties
(Differences from parametrized curves:

- The latter has orientation
- A regular curve may have several branches

The trace of

- A simple regular parametrized curve may not be a (non-self-intersecting) regular curve.

Ex. 4 Let \( \mathbf{a}(t) = (t^3 - 4t, t^2 - 4, 0) \), \(-2 < t < 3\)

Self intersection if \( t \in \mathbb{R} \), but simple \(-2 < t < 3\)

Its trace is \( \bigcirc \) and has no parametrization near \((0, 0)\).

Its extension

\[ \mathbf{x}(t, s) = (t^3 - 4t, t^2 - 4, s), \quad -2 < t < 3, \quad 0 < s < 1 \]

is an example that the trace of a non-self-intersecting regular parametrized surface may not be a regular surface.

(later)
EX 5 (Surface of revolution)

Let $C$ be a connected regular curve on the half $xz$ plane $\Sigma = \{(x,0,z) : x > 0, z \in \mathbb{R}\}$.

Let $C : \gamma = f(t) > 0$ \quad $z = g(t)$, \quad $a \leq t \leq b$, \quad $f, g \in C^\infty$

be a parametrization for $C$.

Rotate it about $z$-axis to get a surface $S$.

One parametrization is

\[ x = f(t) \cos \theta \]
\[ y = f(t) \sin \theta \]
\[ z = g(t). \]

defined in $U_1 = \{(0,t) : 0 < \theta < 2\pi, \quad a \leq t \leq b\} \subset \mathbb{R}^2$.

Its trace $S_1$ does not meet $\Sigma_0 = \Sigma \cup z$-axis.

The same formula defined in

\[ U_2 = \{(t,0) : -\pi < \theta < \pi, \quad a \leq t \leq b\}, \]

gives another parametrization to cover $S$.

Is $S$ a regular surface?

$\varphi : U_1 \rightarrow S_1 \subset (\mathbb{R}^3 \setminus \Sigma_0)$ is clearly smooth, and

\[ d\varphi = \begin{bmatrix} f'c & f's & f' \\ g's & g' & 0 \end{bmatrix} \text{ has rank 2}. \]

To prove that $\varphi$ is $1-1$ and its inverse is continuous,

For given $(x,y,z) \in S_1$, let $r = \sqrt{x^2 + y^2} = f(t)$.

Being a regular curve, the map

\[ (x,y,z) \rightarrow (r,z) \rightarrow t \]

exists and is continuous. The map

\[ (x,y,z) \rightarrow \left( \frac{x}{r}, \frac{y}{r} \right) \rightarrow \theta \]

also exists and is continuous.