Prop 4. Let $S$ be a regular surface. Suppose
\[ \tilde{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad p \in \tilde{x}(U) \subset S, \]
is smooth, $1$-$1$, and $d\tilde{x}_q$ is $1$-$1$ for each $q \in U$. Then
\[ \tilde{x}^{-1} : \tilde{x}(U) \to U \]
is continuous, and hence $\tilde{x}$ is a parametrization of $\tilde{x}(U)$.

Note: $1$-$1 \Rightarrow$ existence of $\tilde{x}^{-1}$, not continuity.

Proof. Let $q \in U$. We may assume $\det \frac{\partial (x,y)}{\partial (u,v)}(q) \neq 0$.

Let $\pi(x,y,z) = (x,y)$. Same argument in the proof of Prop 3

$\Rightarrow \exists$ nbhd $V_1$ of $q$, nbhd $V_2$ of $\pi \tilde{x}(q)$, s.t.

$\pi \circ \tilde{x} : V_1 \to V_2$

has a smooth inverse map $\phi$. Then the restriction
\[ \tilde{x}^{-1} : \tilde{x}(V_1) \to V_1 \]
\[ \tilde{x}^{-1} = \phi \circ \pi, \]
composition of continuous functions. Hence $\tilde{x}^{-1}$ is continuous.
Ex 5  upper cone

\[ S : z = \sqrt{x^2 + y^2}, \quad (x, y) \in \mathbb{R}^2 \]

is not a regular surface.

To show this, it is not sufficient to note that \( f(x, y) = \sqrt{x^2 + y^2} \) is not smooth at \((0, 0)\). By definition, we need to show that there is no map \( \tilde{x} : U \subseteq \mathbb{R}^2 \rightarrow S \) at \( p = (0, 0, 0) \in x(U) \), such that \( \tilde{x} \) is smooth, 1-1 from \( U \rightarrow \text{nbhd of } p \) and \( d\tilde{x}_p \) is 1-1 \( \forall q \in U \).

Instead of using definition, we use Prop 3. Then \( S \) near \( p \) is locally a graph of the form

\begin{align*}
(1) & \quad z = \tilde{f}(x, y) \\
(2) & \quad y = g(x, z), \quad \text{or} \\
(3) & \quad x = h(y, z).
\end{align*}

(2) & (3) are impossible: for all \( x, z \), there are 2 \( y \)'s

\[ \forall y, z, \quad \exists 2 x \]s

(1) : then \( f(x, y) = \tilde{f}(x, y) \) only choice,

but then not smooth.
Ex 6. The surface

\[ S = \{ (u, u^2, uv) : u > 0, \forall v \in \mathbb{R}^2 \} \]

is shown in Ex 4 to be a regular surface.

If \((x, y, z) \in S\), then \(x > 0, z > 0\), and

\[ xz = y^2 \]

Is the map \( F : U \to \mathbb{R}^3 \)

\[ F(x, z) = (x, \sqrt{xz}, z) \]

\[ U = \{ (x, z) \in \mathbb{R}^2 : x > 0, z > 0 \} \]

a parametrization of \( S_+ = F(U) \subset S \)?

Check:

(i) Smooth \( \text{ok} \)

(ii) 1-1 = \( F(x, z) = F(x', z') \) then \((x, z) = (x', z') \) \( \text{ok} \)

(iii) \( dF = \begin{pmatrix} x & 0 \\ \frac{xz}{2x} & \frac{1}{2x} \\ -z & 1 \end{pmatrix} \) \( 1-1 \)

Thus \( F \) is a parametrization of \( S_+ \).

No need to check \( F^{-1} \) continuous, although easy.

How about \( G : U \to \mathbb{R}^3 \)

\[ G(s, t) = (st, s^2t, t^3) \]?
\[
\begin{cases}
 u^3 = s^4 t \\
 u^2 v = s^3 t^2 \\
 u v^2 = t^3 \\
 u = s^{\frac{3}{2}} t^{\frac{1}{2}}, \\
 v = \frac{u^2}{s^2 t^2} = s^{-\frac{3}{2}} t^{\frac{1}{2}}
\end{cases}
\]

Hence \( G(U) = S^+ \)

Smooth ok

1-1 ok

\[
dG = \begin{pmatrix}
4 s^3 t & s^2 \\
2 s^2 t^2 & 2 s t \\
0 & 3 t^2
\end{pmatrix}, \quad 1-1
\]

\( G^1 \) is continuous by Prop 4.
§ 2.3 Smooth functions on surfaces

Goal: To define smooth functions on surfaces, and smooth maps between surfaces, independently of choice of coordinates.

Prop 1 Let $p$ be a pt of a regular surface $S$, and let

$$\bar{x}: U \subset \mathbb{R}^2 \to S, \quad \bar{y}: V \subset \mathbb{R}^2 \to S$$

be two parametrizations of $S$ such that $p \in W = \bar{x}(U) \cap \bar{y}(V)$. Then the map

$$h = (\bar{x})^{-1} \circ \bar{y}: \bar{y}^{-1}(W) \subset V \to \bar{x}^{-1}(W) \subset U$$

is a diffeomorphism, i.e., both $h$ and $h^{-1}$ are smooth.

Rk 1. Being compositions of homeomorphisms, $h$ is also a homeomorphism. (Using Property 2 of Defn of regular surfaces in a vital way.)

2. But "$(\bar{x})^{-1} \in C^\infty(W)$" has no meaning yet.

3. It suffices to show $h \in C^\infty$

$h^{-1} \in C^\infty$ follows by switching $\bar{x}$, $\bar{y}$. 
Idea: Extend \( \overline{X} \) to a map \( \overline{F} \) between 3D regions s.t. \( \overline{F}^{-1} \) is smooth

\[
\begin{align*}
\overline{X} : & \ (u,v) \in U \rightarrow (x,y,z) \in \mathbb{R}^3 \\
\overline{X}_u \& \overline{X}_v & \text{ are } 1, \text{ indep.} \\
& \begin{bmatrix}
\overline{X}_u \\
\overline{X}_v \\
\end{bmatrix}
\text{ has rank } 2,
\end{align*}
\]

one of its 2×2 submatrices has nonzero determinant.

Suppose \( \begin{vmatrix}
\overline{X}_u & \overline{X}_v \\
\overline{Y}_u & \overline{Y}_v \\
\end{vmatrix} \neq 0 \)

\[
\Pi \circ \overline{X} : \ (u,v) \rightarrow (x,y,z) \rightarrow (x,y)
\]

Extend \( \overline{X} \) to \( \overline{F} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \)

\[
\overline{F}(u,v,t) = (x,y,z+t) = \overline{X}(u,v) + (0,0,t)
\]

\[
\begin{vmatrix}
\overline{X}_u & \overline{X}_v & 0 \\
\overline{Y}_u & \overline{Y}_v & 0 \\
\overline{Z}_u & \overline{Z}_v & 1 \end{vmatrix} = \begin{vmatrix}
\overline{X}_u & \overline{X}_v \\
\overline{Y}_u & \overline{Y}_v \\
\overline{Z}_u & \overline{Z}_v \\
\end{vmatrix} \neq 0
\]

By IFT, \( \exists \) nbhd \( V_1 \) of \((q,0)\), nbhd \( M \) of \( p \in \mathbb{R}^2 \),

such that \( \overline{F} : V_1 \rightarrow M \)

has a smooth inverse.

Choose nbhd \( N \subset V \) of \( r \) s.t. \( \overline{y}(N) \subset M \).

Then \( h|N = \overline{F}^{-1} \overline{y}^{-1} |N \)

Composition of smooth fns. Hence \( h|N \) is smooth,

Since \( q \in \overline{X}^{-1}(W) \) is arbitrary, \( h : \overline{y}^{-1}(W) \rightarrow \overline{X}^{-1}(W) \) is smooth. \( \Box \)
Definition: Let \( V \) be an open subset of a regular surface \( S \).

A function \( f: V \to \mathbb{R} \) is differentiable at \( p \in V \) if for some parametrization \( \tilde{x}: U \subset \mathbb{R}^2 \to S \) with \( p \in \tilde{x}(U) \setminus V \), the composition \( f \circ \tilde{x}: U \to \mathbb{R} \) is smooth. It is differentiable in \( V \) if it is differentiable at all \( p \) of \( V \).

Remark: This definition is independent of choice of \( \tilde{x}: W \to S \).

If \( y: W \to S \) is another parametrization, then

\[ f \circ y = (f \circ \tilde{x}) \circ (\tilde{x}^{-1} \circ y) \]

is also smooth, smooth by Prop 1.

2. Similarly for \( f: V \to \mathbb{R}^m \) vector-valued function.

Example: If a regular surface \( S \) is inside an open set \( U \subset \mathbb{R}^3 \), then for any smooth function \( f: V \to \mathbb{R} \), the restriction \( f|_S: S \to \mathbb{R} \) is differentiable.

Reason: Any parametrization \( \tilde{x}: U \subset \mathbb{R}^2 \to S \) maps \( p \),

\( f \circ \tilde{x}: U \to \mathbb{R} \) is composition of smooth functions,

\( f|_S \circ \tilde{x} = f \circ \tilde{x} \).