Remarks on Inverse Function Theorem:

RK1 If $F \in C^k$, $k \in \mathbb{N}$, then $F^{-1} \in C^k$:

Denote $DF(x) = dF_x$

If $F \in C^k(V;W)$, $k \geq 2$, then $DF(y) \in C^{k-1}(V; \mathbb{R}^{nxn})$

$F^{-1} \in C^k(W;V)$, $DF^{-1}(y) = (DF(x)^{-1} \in C^{k-1}(W;\mathbb{R}^{nxn})$

$\Rightarrow F^{-1} \in C^k$

RK2 Instead of $\min |F(x)-y|^2$, $x = F^{-1}(y)$ can be found as the unique fixed pt of the contraction map

$T(x) = T_y(x) = y - g(x)$ in $B = B_r(p)$

$|T(x) - T(x')| = |g(x) - g(x')| \leq \frac{1}{2} |x - x'|$

If $x = T(x) = y - g(x)$,

then $y = x + g(x) = F(x)$.

This argument works in a Banach space.
In particular, fix \( t = a \)

\[
h(u,v) = g(u,v,a)
\]

\( \in C^1(W) : W_i = \{(u,v) : (u,v,a) \in W\} \)

\( (x,y,z) = (u,v, h(u,v)) \in V \) satisfies \( f(x,y,z) = a \)

Conversely, \( F(x,y,z) = a \implies z = h(x,y) \)

The map \( \Phi : (u,v) \in W_i \rightarrow (u,v,h(u,v)) \subset V \)

\( \Phi(W_i) = V \cap f^{-1}(a) \)

\( V \cap f^{-1}(a) \) is a graph & hence regular surface.

\( p \in V \) is arbitrary. \( \implies f^{-1}(a) \) is a regular surface.

\[\Box\]
Ex3 Hyperboloids

\[ S_2 = -x^2 - y^2 + z^2 = 1 \quad \text{has 2 sheets} \]

\[ S_1 = x^2 + y^2 - z^2 = 1 \quad \text{has 1 sheet} \]

Both are regular surfaces.

\[ S_2 = f(x, y, z) = x^2 + y^2 - z^2 + 1 \]

\[ S_1 = g = x^2 + y^2 - z^2 - 1 \]

\[ \nabla f = (2x, 2y, -2z) \neq 0 \quad \text{on } S_1, S_2 \]

\[ S_0: x^2 + y^2 - z^2 = 0 \quad \text{cone} \]

\[ \nabla f = 0 \quad \text{at origin } \in S_0 \]

**Rk** \[ S_2 \] is not connected;

(0,0,1) cannot be joined to (0,0,-1) by a continuous curve on \( S_2 \).

But it is still a regular surface.
Let $C$ be a circle on $yz$ plane centered at $(a,0)$ with radius $r$, $0 < r < a$.

The surface of revolution obtained by rotating $C$ about $z$-axis is a torus.

A parametrization:
\[
\begin{align*}
    z &= r \cos \alpha, & 0 \leq \alpha \leq 2\pi \\
    x &= (a + r \sin \theta) \cos \alpha \\
    y &= (a + r \sin \theta) \sin \alpha, & 0 \leq \theta \leq 2\pi
\end{align*}
\]

Cut at $\theta = 0, \pi, \alpha = 0, \pi$.

Alternatively, $f(x,y,z) \in T$ if
\[
f(x,y,z) = (\sqrt{x^2 + y^2} - a)^2 + z^2 = r^2 \quad \text{(here we used } r < a)\]

Is $r^2$ a regular value of $f$?

\[
f_z = 2z, \quad f_x = 2(\sqrt{x^2 + y^2} - a) \cdot \frac{x}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{2(\sqrt{x^2 + y^2} - a)y}{\sqrt{x^2 + y^2}}
\]

$f$ is smooth if $(x,y) \neq (0,0)$.
When \((x, y) \neq (0, 0)\), \(\nabla f = 0\) iff
\[ z = 0, \quad (\sqrt{x^2 + y^2} - a)x = (\sqrt{x^2 + y^2} - a)y = 0 \]
one of \(x, y \neq 0\), \(\Rightarrow\) \(\sqrt{x^2 + y^2} - a = 0\)
\(\Rightarrow z = \pm r\) since \(f = r^2\)

Thus the torus is a regular surface.

Prop 3... local converse of Prop 1: Any regular graph is locally a graph.

Prop 3
Let \(S \subset \mathbb{R}^2\) be a regular surface \& \(p \in S\). Then

\(\exists\) nbhd \(V\) of \(p\) in \(S\) such that \(V\) is the graph of

a smooth function of the form \(z = f(x, y),\ y = g(x, z),\ z = h(y, z)\).

proof
Let \(\vec{X} : U \subset \mathbb{R}^2 \to S\) be a parametrization

of \(S\) in \(p\), \(\vec{X} = (x, y, z),\ (u, v) \in U\)

Being regular, \(\frac{\partial \vec{X}}{\partial u} \quad \& \quad \frac{\partial \vec{X}}{\partial v}\) are 1, independent.

One of the sub matrices

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}, \quad \frac{\partial (y, z)}{\partial (u, v)} = \frac{\partial (x, z)}{\partial (u, v)}
\]

has nonzero determinant at \(q = \vec{X}^{-1}(p)\)

Assume \(\det \frac{\partial (x, y)}{\partial (u, v)} \neq 0\) at \(q\).
i.e., the projection of 2 tangent vectors $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial u}$ on $xy$-plane are still linearly indep.

Let $\Pi(x, y, z) = (x, y)$ be the projection $\mathbb{R}^3 \to \mathbb{R}^2_{x,y}$

The map $\Pi \circ \bar{X} : (u, v) \mapsto (x(u, v), y(u, v))$

has invertible differential at $\bar{x}$

$$d(\Pi \circ \bar{X})_{\bar{x}} = \frac{\partial (x, y)}{\partial (u, v)} |_{\bar{x}}$$

By Inverse Function Theorem, $\exists$ nbhd $V_1 \subset U$ of $\bar{x}$, $\exists$ nbhd $V_2$ of $\Pi(p)$ such that $\Pi \circ \bar{X} : V_1 \to V_2$

has a smooth inverse map $\bar{\phi}$. Let $V = \bar{z}(V_1)$. Then $V$ is a graph of

$$z = x_3(\bar{\phi}(x, y))$$

over $V_2$. The other 2 cases $\det \frac{\partial (y, z)}{\partial (u, v)} \neq 0$, $\det \frac{\partial (x, z)}{\partial (u, v)} \neq 0$ similar. $\square$