Actually we cannot use Prop 4, which assumes that S is a regular surface, and we don't know it yet.

To show $F'$ continuous, suppose $x > 0$

$$F(x, y) = (x^2, x^2y, xy^2) = (u, v, w), \quad u > 0$$

then

$$x = u^{\frac{1}{3}}$$
$$y = u^{\frac{2}{3}}$$

$$F'(u, v, w) = \left( u^{\frac{1}{3}}, u^{\frac{2}{3}}v \right)$$

is a continuous function on S.
$U = \{ (\theta, \phi) : 0 < \theta < \pi, \; 0 < \phi < 2\pi \}$

$\mathbb{S}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$

Its image is $S^2$ without semicircle $\phi = 0$

$\theta \mapsto (\sin \theta, 0, \cos \theta), \; \quad 0 \leq \theta \leq \pi$

including north/south poles

We need another map to cover $S^2$

For example, removing the semicircle from $(0, 1, 0)$ to $(0, -1, 0)$ through $(-1, 0, 0)$, i.e.

$\theta = \frac{\pi}{2}, \quad \frac{\pi}{2} \leq \phi \leq \frac{3\pi}{2}$
**Inverse Function Theorem:**

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$, and at $p \in U$, $dF_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then $\exists$ nbhd $V$ of $p$ in $U$ and nbhd $W$ of $F(p)$ in $\mathbb{R}^n$ s.t.

\[ F : V \to W \text{ is invertible, and }\]

\[ F^{-1} : W \to V \text{ is } C^1. \]

\[ \begin{array}{c}
V \\
p \\
\hline
\end{array} \quad F \quad \begin{array}{c}
\emptyset \\
W \\
\hline
q = F(p) \end{array} \]

By chain rule, since $id = F^{-1} \circ F$,

\[ I = dF_q \cdot dF_p = (dF_p)'. \]

**Proof**

Let $L = dF_p$. We may assume $L = L_n$ by considering $\tilde{F}(x) = L^{-1}F(x)$.

Let $g(x) = F(x) - x$, $dg_p = 0$. \[ \Rightarrow \exists r > 0 \text{ s.t. } \]

\[ B = \overline{B_r(p)} \subset U, \quad \sup_{B_{1/r}} \left| \frac{\partial g_i}{\partial x_j} \right| < \frac{1}{2n^2}, \quad \det dF(x) \neq 0 \quad \forall x \in B. \]

\[ \Rightarrow \quad |g(x) - g(y)| \leq \frac{1}{2} |x - y| \quad \forall x, y \in B \]

\[ \Rightarrow \quad \frac{1}{2} |x - y| \leq |f(x) - f(y)| \]

Hence $f : B \to f(B)$ is 1-1.

$f^{-1} : f(B) \to B$ satisfies

\[ |f^{-1}(u) - f^{-1}(v)| \leq 2 |u - v| \quad \forall u, v \]

is Lipschitz continuous.
Let $S = \partial B_r(p)$, compact, $f(S)$ compact $\not\subset f(p)$

Let $d = \text{dist}(f(S), f(p)) > 0$

Let $W = B_{\frac{d}{2}}(f(p))$ open

Claim: $W \subset f(B_r(p))$

pf. \[ y \in W, \quad h(y) = |f(x) - y|^2, \quad x \in B \]

\[ \min_{B} h(x) = h(x_0), \]

$x_0 \not\in S$, otherwise $h(x_0) \geq d^2 > h(p)$

$x_0 \in S$, $Dh(x_0) = 0$, i.e.

\[ (f(x_0) - y_i) \cdot \frac{\partial f_i}{\partial x_j} (x_0) = 0 \quad \forall j \]

As $\det Df(x_0) \neq 0$, $f_i(x_0) - y_i = 0$, $f(x_0) = y$.

Let $V = F^{-1}(W)$, open

$F = V \to W$ 1-1, onto, invertible

$F^{-1}$ continuous.

Claim $F^{-1} \in C^1$

pf. \[ u, v \in W, \quad x = f^{-1}(u), y = f^{-1}(v), \]

$v \to u$

\[ f(y) - f(x) - Df_x (y-x) = o(1) |y-x| \]

where $o(1) \to 0 \text{ as } y-x \to 0$

Let $A = Df(x)$.

\[ v - u - A (f'(v) - f'(u)) = o(1) |v-u| \]

using $\triangleright$

\[ f'(u) - f'(v) - A^t (v-u) = o(1) |v-u| \]

$\square$
Prop 1 was for graphs

Prop 2 is concerned with level sets.

Prop 2  If \( f: U \subset \mathbb{R}^3 \to \mathbb{R} \) is smooth and \( a \in f(U) \) is a regular value of \( f \), then
\[
 f^{-1}(a) = \{ (x,y,z) \in U : f(x,y,z) = a \}
\]
is a regular surface.

What is a regular value?

Defn  Given a smooth map \( F: U \subset \mathbb{R}^n \to \mathbb{R}^m \),

- A pt \( p \in U \) is a critical pt of \( F \) if the differential \( dF_p : U^n \to \mathbb{R}^m \) is not onto. Otherwise a regular pt.
- A pt \( q \in F(U) \) is a critical value if one of its preimage \( p \in f^{-1}(q) \) is a critical pt.

Otherwise it is a regular value.

Above: If \( m = 1 \), scalar function,
\[
 dF_p = \nabla F(p), \text{ critical if } \nabla F(p) = 0
\]
\[
 m = n = 1, \quad dF_p : = F'(p)
\]

In Prop 2, every pt in \( f^{-1}(a) \) is a regular pt off.
Proof of Prop 2

Let \( p = (x_0, y_0, z_0) \in f^{-1}(a) \), i.e., \( f(p) = a \).

If \( a \) is a regular value \( \exists p \) is a regular pt,

\[
\text{df}_p = \nabla f(p) : \mathbb{R}^3 \rightarrow \mathbb{R} \text{ is onto.}
\]

i.e. \( \nabla f(p) \neq 0 \)

By changing coordinates, we may assume \( f_z(p) \neq 0 \).

Define \( F : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)

\[
F(x, y, z) = (x, y, f(x, y, z))
\]

\[
dF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}
\]

\[
\text{det}(dF_p) = f_z \neq 0, \quad dF_p \text{ is invertible.}
\]

By Inverse Function Theorem,

\( \exists \) nbhd \( V \) of \( p \), nbhd \( W \) of \( F(p) \)

\[
\begin{array}{ccc}
& & F \\
\uparrow & \swarrow & \\
V & \rightarrow & W
\end{array}
\]

Such that \( F : V \rightarrow W \) is invertible,

\( F^{-1} : W \rightarrow V \) is differentiable.

If \( F(x, y, z) = (u, v, t) \in W \), \( (x, y, z) = F^{-1}(u, v, t) \)

then \( x = u, \ y = v, \ z = g(u, v, t) \in C^1(W) \).
In particular, fix \( t = a \)

\[ h(u,v) = g(u,v,a) \in C'(W_1): W_1 = \{(u,v): (u,v,a) \in W\} \]

\((x,y,z) = (u,v, h(u,v)) \in V\) satisfies \( f(x,y,z) = a \)

Conversely, \( f(x,y,z) = a \Rightarrow z = h(x,y) \)

The map \( \Phi: (u,v) \in W_1 \rightarrow (u,v,h(u,v)) \in V \)

\[ \Phi(W_1) = V \cap f^{-1}(a) \]

\( V \cap f^{-1}(a) \) is a graph & hence regular surface.

\( p \in V \) is arbitrary. \( \Rightarrow f^{-1}(a) \) is a regular surface.

\( \square \)