Defn. A subset \( S \subset \mathbb{R}^3 \) is a regular surface if, for each \( q \in S \), there is a nbhd \( V \subset \mathbb{R}^3 \) of \( q \), an open \( U \) of \( \mathbb{R}^2 \), and a map \( \overline{x} : U \to V \setminus S \) such that

1. \( \overline{x} \) is smooth as a map \( U \to \mathbb{R}^3 \)
2. \( \overline{x} \) is a homeomorphism from \( U \) onto \( V \setminus S \)
3. \( \forall p \in U \), the differential \( d\overline{x}_p : \mathbb{R}^2 \to \mathbb{R}^3 \) is 1-1 (Regularity)

Rk. \( V \) may not be a nbhd of \( V \setminus S \).

\[ \overline{x} \] ... parametrization of \( V \setminus S \)
\[ \overline{x}^{-1} \] ... local coordinates of \( V \setminus S \)

\[ \overline{x}(u,v) = (x(u,v), y(u,v), z(u,v)) \]

Condition 2 avoids self-intersection of \( S \)
Condition 3 allows us to define tangent plane at \( q = \overline{x}(p) \)
Differential

**Defn.** Let \( F: U \subset \mathbb{R}^n \to \mathbb{R}^m \) be a smooth map.

\( \forall p \in U, \quad dF_p : \mathbb{R}^n \to \mathbb{R}^m \) is the linear map defined by

\[
dF_p(w) = \frac{d}{dt} \bigg|_{t=0} F(p + t \alpha(t))
\]

where \( \alpha(t) \) is any curve \( \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^n \) such that

\[
d(0) = p, \quad \alpha'(0) = w.
\]

**Rk.** \( dF_p \) sends a (tangent) vector at \( p \) to a tangent vector at \( F(p) \).

It is independent of choice of \( \alpha(t) \).

It corresponds to a \( m \times n \) matrix:

If \( F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \) for \( x \in U \subset \mathbb{R}^n \)

\[
dF_p(w) = dF_p(w_1 e_1 + \cdots + w_n e_n) = w_1 dF_p(e_1) + \cdots + w_n dF_p(e_n)
\]

Let \( e_1, \ldots, e_n \) be standard basis of \( \mathbb{R}^n \),

\[
dF_p = \begin{bmatrix} dF_p(e_1) & dF_p(e_2) & \cdots & dF_p(e_n) \end{bmatrix} \quad \text{and} \quad dF_p(w) = (dF_p(w)^T)^T
\]
To compute \( \mathrm{d}F_p(e_j) \), choose \( \alpha(t) = p + te_j \), straight line

\[
\mathrm{d}F_p(e_j) = \left. \frac{d}{dt} \right|_{t=0} (f_1, \ldots, f_m)(p + te_j)
\]

\[
= \left( \frac{\partial f_1}{\partial x_j}, \ldots, \frac{\partial f_m}{\partial x_j} \right)(p)
\]

Thus

\[
\mathrm{d}F_p(e_j) = \frac{\partial F}{\partial x_j}(p)
\]

\[
(dF)_{ij} = \frac{\partial f_i}{\partial x_j}, \quad dF_p = \begin{bmatrix} \frac{\partial F^T}{\partial x_1} \\ \vdots \\ \frac{\partial F^T}{\partial x_m} \end{bmatrix}_p
\]

**Example**

\( F : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)

\( F(x, y) = (x^2, xy, xy^2) \)

\[
dF = \begin{bmatrix} \frac{\partial F^T}{\partial x_1} \\ \vdots \\ \frac{\partial F^T}{\partial x_m} \end{bmatrix} = \begin{pmatrix} 3x^2 & 0 \\ 2xy & x^2 \\ y^2 & 2xy \end{pmatrix}
\]

\[
dF(1,1)(2,3) = \left(\begin{pmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}^T \right) = (6, 7, 8)
\]

**Chain rule for maps**:

\[
\begin{array}{ccc}
\mathbb{R}^n & \overset{F}{\rightarrow} & \mathbb{R}^m \\
\psi_p & \quad & \psi_{F(p)} \\
\mathbb{R}^k & \overset{G}{\rightarrow} & \mathbb{R}^k \\
\end{array}
\]

\[
d(G \circ F)_p = dG_{F(p)} \circ dF_p
\]
The regularity condition $dF_p$ is $1-1$ means that

(i) If $dF_p(w) = 0$, then $w = 0$

$w = (w_1, \ldots, w_n)$,

$$dF_p(w) = \begin{bmatrix} \frac{\partial F^T}{\partial x_1} & \cdots & \frac{\partial F^T}{\partial x_n} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= w_1 \frac{\partial F^T}{\partial x_1} + \cdots + w_n \frac{\partial F^T}{\partial x_n}$$

Thus (i) is equivalent to

(ii) $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$ are linearly independent.

Each $dF_p(w)$ is a tangent vector of $F(U)$ at $F(p)$

The set $\{dF_p(w) : w \in \mathbb{R}^n\}$ is spanned by

$$\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$$

Thus their span is $n$-dimensional. (iii)
Example again

\[ dF_{(1,1)}(e_1) = (3, 2, 1) \]

\[ dF_{(1,1)}(e_2) = (0, 1, 2) \]

They are linearly independent and span the tangent plane of the trace of \( F \) at \( F(1,1) = (1,1,1) \).

At a general point \((u,v)\)

\[ dF_{(u,v)} = \begin{pmatrix} 3u^2 & 0 \\ 2uv & u^2 \\ v^2 & 2uv \end{pmatrix} \]

Linear dependence \( \Rightarrow \) \( u = 0 \)

\[ dF_p \text{ is not regular at } (u,v) = (0,1) \quad dF_{(0,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

regular at \( (u,v) = (1,0) \), \( dF_{(1,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \).

\[ \begin{cases} \text{Linear dep } \Rightarrow u=0 \text{ by looking at 1st component.} \\ \text{If } u=0, \text{ 2nd vect} \theta = \overrightarrow{0}. \Rightarrow \text{linear dependence.} \end{cases} \]

**Question.** If \( U = \{(u,v) \in \mathbb{R}^2, u > 0\} \) and \( S = F(U) \), is \( S \) a regular surface?

Condition 1: Smooth OK, Condition 3 regular checked.

Condition 2: If \( F(u,v) = F(x,y) \), then \( 3u^2 = 3\overrightarrow{x^2} \Rightarrow u = x \)

\( 2uv = 2\overrightarrow{xy} \Rightarrow v = y \) then \( 0, \overrightarrow{F} \) is a inverse \( F^{-1} \).

Is \( F^{-1} \) continuous? Yes by Proposition 4 later.
Prop 1. The graph of a smooth function \( f : U \times \mathbb{R} \to \mathbb{R} \), i.e.,
\[
S = \{ (u, v, f(u, v)) : (u, v) \in U \}
\]
is a regular surface.

Proof: The map
\[
\varphi : U \times \mathbb{R}^2 \to \mathbb{R}^3,
\]
\[
\varphi(u, v) = (u, v, f(u, v))
\]
is clearly smooth. It is a homeomorphism with continuous inverse (the projection)
\[
\varphi : S \to U,
\]
\[
\varphi(x, y, z) = (x, y)
\]
Finally,
\[
\frac{\partial F}{\partial u} = \begin{bmatrix} \frac{\partial x}{\partial u} \end{bmatrix} \quad \frac{\partial x}{\partial v} \begin{bmatrix} \frac{\partial x}{\partial v} \end{bmatrix}^T
\]
\[
\frac{\partial F}{\partial u} = (1, 0, \frac{\partial f}{\partial u}), \quad \frac{\partial x}{\partial v} = (0, 1, \frac{\partial f}{\partial v})
\]
are clearly independent for \((u, v) \in U\).
Ex2  The unit sphere $S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$

is a regular surface.

Remark  One open set is not enough, since $S^2$ cannot
be homeomorphic to an open set $U$ of $\mathbb{R}^2$.

Proof.  $S^2$ is contained in $\bigcup_{j=1}^{6} V_j$, where

$$V_j = \{ x_j > 0 \}, \quad V_j + 3 = \{ x_j < 0 \}, \quad j = 1, 2, 3.$$  

Let $S_j = S^2 \cap V_j$. Each $S_j$ is a graph. For example,$$S_5 = S^2 \cap V_5 = \{ (x_1, f(x_1, x_3), x_3) : (x_1, x_3) \in U \},$$

where $f(x_1, x_3) = -\sqrt{1 - x_1^2 - x_3^2}$ is defined in the
unit disk $U; \ x_1^2 + x_3^2 < 1$.

Each $S_j$ is regular surface, by Prop 1.

Since each of $S^2 \cap V_j$ is a regular surface, so is $S^2$.

Rk. We have used: If $S \subset \bigcup_{j=1}^{N} V_j$, each of $S \cap V_j$ is
a regular surface, then $S$ is a regular surface.

However, if $S = \bigcup_{j=1}^{N} S_j$, each of $S_j$ is a regular
surface, $S$ may not be a regular surface, due
to possible self-intersection.