§1.7 Global properties of plane curves

**Def.** A curve \( \alpha: I = [a, b] \to \mathbb{R}^2 \) is

- **regular**, if \( \alpha'(t) \neq 0 \ \forall t \in I \) (not automatic if \( \alpha \in C^1 \))
- **closed**, if \( \alpha(a) = \alpha(b) \) and \( \alpha^{(k)}(a) = \alpha^{(k)}(b) \ \forall k = 0, 1, 2, \ldots \)
- **simple**, if it is non-self-intersecting, i.e.,
  - if \( t_1, t_2 \in [a, b], \ t_1 \neq t_2 \), then \( \alpha(t_1) \neq \alpha(t_2) \)

\( \bigcirc \) non-simple, closed.

§5.7 Jordan Curve Theorem

Let \( C \) be a simple closed curve in \( \mathbb{R}^2 \).

Then its complement \( \mathbb{R}^2 \setminus C \) consists of exactly 2 connected components \( \Omega_0, \Omega_1 \).

- \( \Omega_0 \) ... bounded, called the interior of \( C \)
- \( \Omega_1 \) unbounded, exterior

and \( \partial \Omega_0 = \partial \Omega_1 = C \).

\( C = \alpha: I \to \mathbb{R}^2 \) is called **positively oriented** if its interior is on left side when we move along \( C \).

We call \( \text{Area}(\Omega_0) \) the area enclosed by \( C \).

**Rk** The version \( C \in C^1 \) is in §5.7.

A curve is always \( C^0 \).

In the direction of increasing parameters.
A. Isoperimetric inequality

The problem: Of all simple closed curves in the plane with a given length $l$, which one bounds the largest area?

It means the area of its interior.

The answer is a circle.

Old arguments assume the existence of such an area maximizer, but it needs to be proved.

\[ \text{area formula} \quad \text{Let } \mathbf{d}(t) = (x(t), y(t)), \quad t \in [a, b], \text{ be a piecewise } C^1 \text{ pos. oriented simple closed curve in } \mathbb{R}^2. \text{ The area it bounds is} \]

\[ A = \int_a^b x \, y' \, dt = -\int_a^b x' \, y \, dt = \frac{1}{2} \int_a^b (xy' - x'y) \, dt \]

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Theorem Let $C$ be a simple closed curve with length $l$, and let $A$ be the area of the region bounded by $C$. Then

\[ 4\pi A \leq l^2 \]

and equality holds iff $C$ is a circle.
proof. Let \( E \& E' \) be 2 parallel lines enclosing \( C \). Move them until they first meet \( C \). Call the new parallel lines \( L \& L' \).

Let \( P \& P' \) be the northeast points where \( L \& L' \) meet \( C \).

(there could be other points where \( L \& L' \) meet \( C \))

parametrize \( C \) by arc length \( s \):

\[
\alpha(s) : \ [0, l] \rightarrow \mathbb{R}^2 \\
= (x(s), y(s))
\]

such that \( P = \alpha(0) \) and \( P' = \alpha(s_1) \), & \( C \) is p.s. oriented.

Let \( S' \) be a circle which is tangent to both \( L \& L' \) and does not meet \( C \). We may a coordinate system with the center of \( S' \) being the origin, and the \( y \)-axis parallel to \( L, L' \), \( L : x = r \), \( L' : x = -r \).
we may parametrize $S^1$ by

$$\overline{a}(s) = (\overline{x}(s), \overline{y}(s)) \quad \text{with} \quad \overline{x}(s) = x(s), \quad s \in [0, l].$$

The choice of $\overline{y}(s)$ is

\[
\begin{cases} 
\overline{y}(s) \geq 0 & 0 < s < s_1, \\
\overline{y}(s) \leq 0 & s_1 < s < l
\end{cases}
\]

$\overline{a}(s)$ may move back & forth on $S^1$

$$A = \int_0^l xy' \, ds, \quad \overline{A} = \text{area}(S^1) = \pi r^2 = -\int_0^l \overline{y} x' \, ds.$$

$$A + \overline{A} = \int_0^l (xy' - \overline{y} x') \, ds$$

Cauchy - Schwarz inequality:

$$ab + cd \leq \sqrt{a^2 + c^2} \cdot \sqrt{b^2 + d^2}; \quad "=\" \text{ holds if } \quad (a, c) \parallel (b, d) \quad \text{iff } \quad ad = bc$$

$$= \int_0^l \sqrt{x^2 + \overline{y}^2} \cdot \sqrt{y^2 + x'^2} \, ds \quad = 1$$

$$= \int_0^l r \, ds = lr$$

Note

$$A + \overline{A} = 2\sqrt{A\overline{A}} = 2r\sqrt{4\pi r}$$

Thus

$$2r\sqrt{4\pi} \leq lr, \quad 4\pi r \leq l^2$$

If equality holds,

$$xx' = -\overline{y}y' \quad \text{for all } s \quad \text{where } a \text{ is } C^1.$$
\[ xx' = -\sqrt{r^2-x^2} \ y' \quad \quad x^2 + y^2 = 1 \]

\[ \frac{x^2 - x'^2}{r^2} = (r^2-x^2)(1-x'^2) = \frac{r^2-x^2 - r^2 x^2 + x^2 x'^2}{r^2} \]

\[ x'^2 = \frac{r^2-x^2}{r^2} \]

\[ \frac{dx}{\sqrt{r^2-x^2}} = \pm \frac{ds}{r} \]

\[ \arcsin \frac{x}{r} = \pm \frac{s-s_0}{r} \]

\[ \frac{x}{r} = \pm \sin \frac{s-s_0}{r} \]

\[ y' = -\frac{x}{\sqrt{r^2-x^2}} \quad x' = \pm \frac{x}{r} = \pm \sin \frac{s-s_0}{r} \]

\[ y = \pm r \cos \frac{s-s_0}{r} + y_0 \]

\[ x^2 + (y-y_0)^2 = r^2. \]

Hence \( C \) is a circle of radius \( r \) if \( \alpha(s) \in C^1(0, \ell) \).

**Rk** It would take some extra work if we only assume \( C \) is piecewise \( C^1 \).