MATH 424 summary

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In the future I plan to either type or scan my lecture notes.

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0.1 Notation

(a, b) means an interval on the real line, with $-\infty \leq a < b \leq +\infty$.

We say $f : I = (a, b) \to \mathbb{R}$ is of class $C^k$, $k \in \mathbb{N}$, denoted as $f \in C^k(I)$, if $f$ has derivatives in $I$ up to order $k$, and $f^{(k)}$ is continuous in $I$.

We say $f \in C^\infty(I)$ if $f \in C^k(I)$ for all $k \in \mathbb{N}$. In this case we say $f$ is smooth.

For $p, q \in \mathbb{R}^n$, the dot product and vector length are

$$p \cdot q = p_1 q_1 + \cdots + p_n q_n, \quad |p| = \sqrt{p \cdot p}$$

1 Curves

1.1 Parametrized curve

A parametrized curve in $\mathbb{R}^n$, $n \geq 2$, is

$$\alpha : I = (a, b) \to \mathbb{R}^n, \quad -\infty \leq a < b \leq +\infty,$$

$$\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t)) \in \mathbb{R}^n.$$ 

We can take closed interval $I = [a, b]$ if $a, b$ are finite.

We say $\alpha \in C^k(I, \mathbb{R}^n)$ if $\alpha_j \in C^k(I)$ for $j = 1, \ldots, n$.

The image set

$$\alpha(I) = \{\alpha(t) : t \in I\}$$

is called the image or trace of $\alpha$. We need $\alpha \in C^0$ so that its trace is connected. This is not good enough since a $C^0$ curve may be dense in $\mathbb{R}^n$. We need $\alpha$ to be at least piecewise $C^1$, with means that there are $a = t_0 < t_1 \cdots t_k = b$ such that

$$\alpha \in C^0(a, b), \quad \alpha \in C^1(t_{j-1}, t_j), \quad \forall j = 1, \ldots, k$$

Examples.

1. $\alpha(t) = \vec{q} + t\vec{p}, \ t \in \mathbb{R}$, and $\vec{q}, \vec{p} \in \mathbb{R}^n$ are constant vectors. This is a straight line in $\mathbb{R}^n$.

2. $\alpha(t) = (a \cos kt, a \sin kt, bt), \ t \in \mathbb{R}, \ a, b, k \in R$ are constants. $a, k > 0$.

   (a) If $b = 0$, this is a circle of radius $a$. 

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(b) If $b \neq 0$, this is a helix, obtained by $\alpha(t) = (a \cos kt, a \sin kt, 0) + (0, 0, bt)$, rotation with constant angular velocity plus translation with constant velocity.

3. $\alpha(t) = (t^2, t^3)$, $t \in \mathbb{R}$. Note that $\alpha(t) \in C^\infty$, but its image has a cusp at $\alpha(0) = (0, 0)$. Indeed, if we consider the unit tangent vector

$$\hat{T}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$$

its limit as $t \to 0_-$ is $(-1, 0)$, while its limit as $t \to 0_+$ is $(1, 0)$. $\hat{T}$ is discontinuous at $t = 0$ although $\alpha \in C^\infty$ because $\alpha'(0) = 0$.

**Definition.** A parametrized curve $\alpha : I = (a, b) \to \mathbb{R}^n$ is regular if $\alpha \in C^1(I)$ and $\alpha'(t) \neq 0$ everywhere in $I$.

When $\alpha'(t_0) \neq 0$, it is a tangent vector to $\alpha(t)$ at $t = t_0$. The tangent line at $\alpha(t_0)$ is

$$\beta(\lambda) = \alpha(t_0) + \lambda \alpha'(t_0), \quad \lambda \in \mathbb{R}$$

It is the first order approximation of the curve, as the first order Taylor expansion of $\alpha(t)$ near $t_0$ is

$$\alpha(t_0 + \lambda) = \alpha(t_0) + \lambda \alpha'(t_0) + o(\lambda) = \beta(\lambda) + o(\lambda), \quad (|\lambda| \ll 1)$$

### 1.2 Reparametrization

Let a curve $\alpha : I = (a, b) \to \mathbb{R}^n$ be given. Suppose

$$f : [c, d] \to [a, b]$$

is $C^1$, 1-1 and onto, and $f' \neq 0$. Then $f$ is strictly monotone. Define

$$\beta : [c, d] \to \mathbb{R}^n, \quad \beta = \alpha \circ f, \quad \beta(\tau) = \alpha(f(\tau))$$

It has the same image. Since

$$\frac{d}{d\tau} \beta(\tau) = \frac{d}{dt} \alpha(f(\tau)) \cdot f'(\tau), \quad f'(\tau) \neq 0,$$

$\beta$ is regular if and only $\alpha$ is regular.

In case $f' > 0$, $\alpha$ and $\beta$ have the same orientation.

In case $f' < 0$, $c > d$ and $\alpha$ and $\beta$ have opposite orientation.

### 1.3 Arc length

The length of $\alpha : [a, b] \in \mathbb{R}^n$ is

$$\int_a^b \left| \frac{d}{dt} \alpha(t) \right| \, dt$$

It is invariant under reparametrization, as

$$\int_c^d \left| \frac{d}{d\tau} \beta(\tau) \right| \, d\tau = \int_c^d \left| \frac{d}{dt} \alpha(f(\tau)) \cdot f'(\tau) \right| \, d\tau$$

$$= \int_a^b \left| \frac{d}{dt} \alpha(t) \right| \, dt$$

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if \( f' > 0 \), by chain rule and change of variable formula for integrals. It is true if \( f' < 0 \) since 
\[
- \int_{a}^{b} f'(t) dt = \int_{a}^{b} f(t) dt.
\]

If \( \alpha(t) \) is regular, we can reparametrize it by arclength: Introduce the arclength parameter
\[
S(t) = \int_{a}^{t} \left| \frac{d\alpha(t)}{dt} \right| dt.
\]

Suppose \( L = \int_{a}^{b} \left| \frac{d\alpha(t)}{dt} \right| dt < \infty \). We have \( S(a) = 0 \), \( S(b) = L \), and \( S'(t) = \left| \frac{d\alpha(t)}{dt} \right| > 0 \).

Thus \( S \in C^1 \) is invertible (here we use in an essential way that \( \alpha \) is regular), and
\[
S^{-1} : [0, L] \to [a, b]
\]

Let \( \beta = \alpha \circ S^{-1} : [0, L] \to \mathbb{R}^n \) and \( t = S^{-1}(s) \). We have
\[
\frac{d\beta(s)}{ds} = \frac{d\alpha(t)}{dt} \cdot \frac{dS^{-1}(t)}{ds} = \frac{d\alpha(t)}{dt} \cdot \frac{1}{dS/dt} = \frac{\alpha'(t)}{|\alpha'(t)|}.
\]

In particular,
\[
\left| \frac{d\beta(s)}{ds} \right| = 1.
\]

We have shown the following.

**Proposition.** Every regular curve can be re-parametrized by its arclength parameter.

**Example.** Circular helix \( \alpha(t) = (a \cos kt, a \sin kt, bt), t \in \mathbb{R} \), with constant \( a, b, k > 0 \).

We have
\[
\alpha'(t) = (-ak \sin kt, ak \cos kt, b),
\]
\[
|\alpha'(t)| = \sqrt{a^2 k^2 + b^2} =: m
\]

Thus
\[
S(t) = \int_{t_0}^{t} m \, dt = m(t - t_0)
\]

If we set \( t_0 = 0 \) and \( s = S(t) = mt \), we have \( t = S^{-1}(s) = s/m \) and
\[
\beta(s) = (a \cos \frac{ks}{m}, a \sin \frac{ks}{m}, \frac{bs}{m})
\]

### 1.4 Curvature

We consider now the second order approximation. Note that, when parametrized by arclength, \( \alpha = \alpha(s) \), we have
\[
|\alpha'(s)|^2 = \alpha'(s) \cdot \alpha'(s) = 1
\]

Taking derivative,
\[
2\alpha'(s) \cdot \alpha''(s) = 0
\]

This says that \( \alpha''(s) \) is orthogonal to \( \alpha'(s) \). This would not be true if \( s \) were not the arclength. Define the **curvature**
\[
k(s) = |\alpha''(s)|
\]

When \( k \neq 0 \), define the **principle normal vector**
\[
n = \frac{\alpha''(s)}{|\alpha''(s)|}
\]