Ex 5. Find the work done by the force \( \mathbf{F}(x) = \begin{cases} \frac{-x^2}{1 \times 10^6}, & x \in \mathbb{R}^3 \\ 0, & x \neq 0 \end{cases} \)

when a particle is moved from \((1,1,1)\) to \((2,3,4)\),
then from \((2,3,4)\) to \((-2,1,3)\), along 2 line segments.

**Sol.** \( \mathbf{F} \) is conservative:

If \( \phi(x) = f(|x|) \) is a radial function,

By chain rule,

\[
\nabla \phi(x) = f'(|x|) \nabla |x| \\
= f'(|x|) \cdot \frac{x}{|x|}
\]

In our case,

\[
\mathbf{F} = -\frac{x}{|x|^3}
\]

\[
\frac{x'}{t} = -\frac{1}{t^5}, \quad f(t) = \frac{1}{8 \cdot t^8}
\]

\[
\phi(x) = \frac{1}{8 \cdot |x|^8}
\]

By Theorem 1

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(-2,1,3) - \phi(1,1,1)
\]

\[
= \frac{1}{8 \cdot 14^4} - \frac{1}{8 \cdot 3^4}
\]
Theorem 2 (Characterization of conservative vector fields)

Let $\mathbf{F}$ be a continuous vector field in an open connected subset $D$ of $\mathbb{R}^n$. The following are equivalent:

(a) $\mathbf{F}$ is conservative in $D$, $\mathbf{F} = \nabla \phi$

(b) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curve $C$

(c) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent, i.e.,

$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ whenever $C_1$ and $C_2$ have same initial and final points.

Remarks

1. A curve $C$ is closed if its initial and final points are the same. The circle on $\int_C$ indicates that $C$ is closed.

2. A set $D$ is connected if any 2 points in $D$ can be joined by a continuous curve in $D$. [Diagrams showing connected and disconnected sets]
Remarks

3). \(D\) is open means every point \(P\) in \(D\) is an interior point, that is, there is a small \(r > 0\) such that 

\[ B(P, r) = \{ x \mid |x - p| < r \} \]

is a subset of \(D\).

The set \(D\) cannot contain a point \(Q\) on its boundary. This ensures that we can take partial derivatives at \(P\).

Ex 6

1). \(D = \mathbb{R}^2\) or \(D = \mathbb{R}^3\)

(This is the case in CLP-4, but Theorem 2 is really valid for our general \(D\).)

2). An open ball (excluding boundary) is open.

A closed ball (including boundary) is not open.

\[ x^2 + y^2 < 9 \]

\[ x^2 + 9 \leq 9 \]

3). A punctured open ball

\[ 0 < x^2 + y^2 < 9 \]
Proof of Theorem 2: 3 parts: (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c) \(\Rightarrow\) (a)

Suppose \( F = \nabla \phi \). Same

(a) \(\Rightarrow\) (b): If \( C \) is any closed curve with initial \( P \) and final point \( Q \), then, by Theorem 1,

\[
\oint_C F \cdot d\mathbf{r} = \phi(Q) - \phi(P) = 0.
\]

(b) \(\Rightarrow\) (c). Suppose \( C_1 \) and \( C_2 \) are two curves that start at \( P \) & end at \( Q \).

\[\begin{aligned}
\text{Let } C_3 &= -C_2 \text{ if } C_2 \text{ given by } \mathbf{r}_2(t), \\
&\quad 0 \leq t \leq 1, \\
\text{then } C_3 \text{ is given by } \mathbf{r}_3(t) &= \mathbf{r}_2(1-t),
\end{aligned}\]

Let \( C = C_1 \cup C_3 \ldots \text{closed curve, } P \rightarrow Q \rightarrow P \)

By (b), \( 0 = \int_C F \cdot d\mathbf{r} \)

But \( \text{RHS} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_3} F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} - \int_{C_2} F \cdot d\mathbf{r} \)

Thus \( \int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r} \).
(c) \Rightarrow (a): We need to find \( \phi \). Motivated by Thm 1, fix \( P_0 \in D \), and define for any \( P \in D \)

\[
\phi(P) = \int_C F \cdot dr
\]

where \( C \) is any curve in \( D \) that starts at \( P_0 \) and ends at \( P \). By (c), \( \phi \) is uniquely defined.

To check: \( \nabla \phi = F \) ??

Consider \( \frac{\partial \phi}{\partial x}(P) \). Fix \( C_p \) any curve from \( P_0 \) to \( P \).

\[
\phi(P + s \hat{x}) = \int_{C_p \cup D_s} F \cdot dr
\]

\( D_s : F(t) = P + st \hat{x}, \quad 0 \leq t \leq 1. \)

Thus

\[
\phi(P + s \hat{x}) = \phi(P) + \int_{D_s} F \cdot dr
\]

\( = \phi(P) + \int_{D_s} F \cdot dx \)

Since \( dy = 0, dz = 0 \) on \( D_s \)

\[
\frac{\partial \phi}{\partial x}(P) \text{ is the limit of } \frac{\phi(P + s \hat{x}) - \phi(P)}{s} = \frac{1}{s} \int_{D_s} F \cdot dx \quad \text{as } s \to 0
\]

Similarly for \( \frac{\partial \phi}{\partial y} \) and \( \frac{\partial \phi}{\partial z} \). \( \square \)