Another way to remember the condition.

Recall \( \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{\hat{i}} & \mathbf{\hat{j}} & \mathbf{\hat{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \)

Def. The curl of a vector field \( \mathbf{F}(x, y, z) \) in \( \mathbb{R}^3 \), denoted as \( \text{curl} \, \mathbf{F} = \nabla \times \mathbf{F} \), is

\[
\text{curl} \, \mathbf{F} = \nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{\hat{i}} & \mathbf{\hat{j}} & \mathbf{\hat{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} = (\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2) \mathbf{\hat{i}} + (\frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3) \mathbf{\hat{j}} + (\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1) \mathbf{\hat{k}}
\]

Rk. i) We treat \( \nabla \) in \( \nabla \times \mathbf{F} \) as a vector.

ii). The condition \( \bigotimes_{3D} \) is the same as \( \nabla \times \mathbf{F} = \mathbf{0} \). Condition \( \bigotimes_{2D} \) is a special case.

iii). Physical meaning: \( \S 4.1 \)
Ex4. Compute curl \( \vec{F} \) for \( \vec{F}(x,y,z) = x^2 y \hat{i} + yz^2 \hat{j} + zx^2 \hat{k} \). Conclude that \( \vec{F} \) is not conservative.

**Soln**

\[
\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x^2 y & yz^2 & zx^2
\end{vmatrix}
\]

\[
= \begin{vmatrix}
yz^2 & zx^2 & \hat{i} \\
x^2 y & \hat{j} & \hat{k}
\end{vmatrix} - \begin{vmatrix}
x^2 y & zx^2 & \hat{i} \\
yz^2 & \hat{j} & \hat{k}
\end{vmatrix} + \begin{vmatrix}
x^2 y & yz^2 & \hat{i} \\
xz^2 & \hat{j} & \hat{k}
\end{vmatrix}
\]

\[
= (0 - 2yz) \hat{i} - (2xz - 0) \hat{j} + (0 - x^2) \hat{k}
\]

\[
= -2yz \hat{i} - 2xz \hat{j} - x^2 \hat{k}
\]

Since curl \( \vec{F} \neq \vec{0} \), \( \vec{F} \) is not conservative.

Ex3a again

\[
\vec{F} = (ay^2, 4xy-z, by+1)
\]

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
ay^2 & 4xy-z & by+1
\end{vmatrix}
\]

\[
= \begin{vmatrix}
ay^2 & 4xy-z & \hat{i} \\
4xy-z & \hat{j} & \hat{k}
\end{vmatrix}
\]

\[
= (b+1, 0-0, 4y-2ay) = \vec{0}
\]

Hence \( b+1=0, \quad 4-2a=0, \quad a=2, \quad b=-1. \)
Ex 5. Consider the vector field

\[ \mathbf{F} = \frac{-y}{x^2+y^2} \mathbf{\hat{i}} + \frac{x}{x^2+y^2} \mathbf{\hat{j}} \]

defined in the domain \( D = \mathbb{R}^2 \setminus \{(0,0)\} \)

(a) Check that \( \text{curl } \mathbf{F} = \mathbf{0} \).

(b) Show that \( \mathbf{F} \) is conservative in the subdomain \( D_1 = \{(x,y) \in \mathbb{R}^2 : x > 0\} \)

i.e., the right half plane.

(c) \( \mathbf{F} \) is also conservative in the upper half plane

\( D_2 = \{(x,y) \in \mathbb{R}^2 : y > 0\} \)

(d) But \( \mathbf{F} \) is not conservative in \( D \).
Rk. i). This is an example of a non-conservative vector field with zero curl.

ii). This example shows that, to say a vector field is conservative, we need to specify its domain.

iii). If a vector field \( \mathbf{F} \) is conservative in \( D \) and \( D_1 \subset D \), then \( \mathbf{F} \) is also conservative in \( D_1 \), because \( \mathbf{F} = \nabla \phi \) in \( D \) implies \( \mathbf{F} = \nabla \phi \) in \( D_1 \).

(iv). The reversed statement is wrong. A conservative vector field in \( D_1 \) may not be conservative in \( D \).

Soln (a). \( \text{curl} \, \mathbf{F} = \left( \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) \right) \hat{k} 

= \left\{ \frac{1}{x^2 + y^2} - \frac{x \cdot 2x}{(x^2 + y^2)^2} + \frac{1}{x^2 + y^2} - \frac{y \cdot 2y}{(x^2 + y^2)^2} \right\} \hat{k} 

= \left\{ \frac{2}{x^2 + y^2} - \frac{2x^2 + 2y^2}{(x^2 + y^2)^2} \right\} \hat{k} = 0 \)
(b). In fact \( \mathbf{F} = \nabla \phi \)

\[ \phi = \Theta + c \quad \text{const} \]

\( \Theta \) ... angle in polar coordinates

\[ -\frac{\pi}{2} < \Theta < \frac{\pi}{2} \quad \text{in } D_1 \]

\[ \Theta = \arctan \frac{y}{x} \]

check:

\[ (\arctan \pm)' = \frac{1}{1 + \pm^2} \]

\[ \frac{\partial}{\partial x} (\arctan \frac{y}{x}) = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \cdot \frac{x}{x^2} = \frac{x^2}{x^2 + y^2}, \quad \frac{-y}{x^2} = \frac{-y}{x^2 + y^2} \]

\[ \frac{\partial}{\partial y} (\arctan \frac{y}{x}) = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \cdot \frac{1}{x} = \frac{1}{x^2 + y^2} \]

\[ \mathbf{F} = \nabla \phi \quad \phi = \Theta + c \quad \text{const} \]

also works.

(C). \( \mathbf{F} = \nabla \Theta \) still works, \( 0 < \Theta < \pi \) in \( D_2 \).

\[ \Theta = \arccot \frac{x}{y} \]

(d). If \( \mathbf{F} = \nabla \phi \) in \( D \), then \( \mathbf{F} = \nabla \phi \) in \( D_1 \) & \( D_2 \)

\[ \phi = \Theta + c_1 \quad \text{in } D_1 \]

\[ \phi = \Theta + c_2 \quad \text{in } D_2 \]

\( \phi = \Theta + c \quad \text{the only candidate} \)

But \( \Theta \) is NOT a (single-valued) continuous function in \( D \)!