\textbf{Ex. 2} \quad \textit{Gravity}

\[ F(r) = -\frac{G M m}{r^3} \hat{r}, \quad r = |\vec{r}| \]

\[ = \nabla \varphi(\vec{r}), \quad \varphi(\vec{r}) = \frac{G M m}{r} \]

\textbf{Check:} \quad \frac{\partial}{\partial x} \frac{G M m}{\sqrt{x^2 + y^2 + z^2}} = -\frac{1}{2} \frac{G M m}{\sqrt{x^2 + y^2 + z^2}} \cdot 2x

\[ = -\frac{G M m}{r^3} x \]

\textbf{Lemma} \quad \text{If} \ \vec{F} \ \text{is a conservative vector field, then}

\text{Flow lines of} \ \vec{F} \ \text{are orthogonal to equi potential surfaces of} \ \varphi.

\textbf{Proof.} \quad \text{Suppose} \ \vec{F}(t) \ \text{is any curve on an}

\text{equi potential surface,}

\[ \varphi(\vec{F}(t)) = C_1 \]

\[ 0 = \frac{\partial}{\partial x} \varphi(\vec{F}) \cdot \frac{dx}{dt} + \frac{\partial}{\partial y} \varphi(\vec{F}) \cdot \frac{dy}{dt} + \frac{\partial}{\partial z} \varphi(\vec{F}) \cdot \frac{dz}{dt} \]

\[ \frac{d}{dt} : = \nabla \varphi(\vec{F}) \cdot \frac{d\vec{F}}{dt} \]

\text{I.e.} \ \vec{F} = \nabla \varphi(\vec{F}) \perp \frac{d\vec{F}}{dt}

\text{Since} \ \vec{F}(t) \ \text{is arbitrary,} \ \vec{F} \ \text{is orthogonal to} \ \varphi = C_1.
Ex 1 again

(a) \( F = (x, y), \ y = \frac{1}{2}(x^2 + y^2) = c \)

(b) \( F = (x, -y), \ y = \frac{1}{2}(x^2 - y^2) = c \)

(c) \( F = (y, x) \)
\[ y = xy = c \]

(d) N/A

Ex 2 again
\[ \psi(F) = \frac{G M m}{r} = \text{const } c \quad \ldots \quad \text{spheres} \]
\[ r = \frac{G M m}{c} \]

orthogonal to \( F \)
We gave 2 reasons why $\mathbf{E}x1(d) \ F = (-y, x)$ is not conservative. we now explore the second:

If $\mathbf{F} = (F_1, F_2)$ is conservative, $\mathbf{F} = \nabla \varphi = (\varphi_x, \varphi_y)$

then $F_1 = \partial_x \varphi = \frac{\partial \varphi}{\partial x}$

$F_2 = \partial_y \varphi = \frac{\partial \varphi}{\partial y}$

It is necessary that $\varphi_{xy} = \varphi_{yx}$, i.e.

$\partial_y F_1 = \partial_x F_2 \quad \bigotimes_{2D}$

In 3D

if $\mathbf{F} = (F_1, F_2, F_3) = \nabla \varphi = (\varphi_x, \varphi_y, \varphi_z)$

then $\varphi_{xy} = \varphi_{yx} \Rightarrow \partial_y F_1 = \partial_x F_2$

$\varphi_{xz} = \varphi_{zx} \Rightarrow \partial_z F_1 = \partial_x F_3$

$\varphi_{yz} = \varphi_{zy} \Rightarrow \partial_z F_2 = \partial_y F_3$

Lemma If $\mathbf{F}$ is conservative, then we have

$\bigotimes_{2D}$ in 2D, and $\bigotimes_{3D}$ in 3D.

Remark. The necessary condition is not sufficient.

There are non-conservative vector fields that satisfy the condition. Examples later.
Ex.3. The vector field
\[ \mathbf{F} = (ay^2, 4xy-z, by+1) \]
is conservative
(a) Find the const a & b.
(b) Find its potential \( \Phi \).

Solv. (a) \( \Phi_{xy} = \partial_y (ay^2) = \partial_x (4xy-z) \)
\[ 2ay = 4y, \quad a=2. \]
\( \Phi_{xz} = \partial_z (ay^2) = \partial_x (by+1) \)
\[ 0 = 0. \]
\( \Phi_{yz} = \partial_z (4xy-z) = \partial_y (by+1) \)
\[ -1 = b, \quad b=1. \]

(b) \( \mathbf{F} = \nabla \Phi, \quad \Phi_x = 2y^2, \quad \Phi_y = 4xy-z, \quad \Phi_z = -y+1 \)
\( \Phi_x = 2y^2, \quad \Phi = \int 2y^2 \, dx = 2xy^2 + g(y,z) \)
\( 4xy-z = \Phi_y = 4xy + g_y, \)
\( g_y = -z, \quad \text{z as const} \)
\( g(y,z) = \int -z \, dy = -yz + h(z) \)
\( -y+1 = \Phi_z = 0, \quad -y + h'(z) \)
\[ h'(z) = 1, \quad h(z) = z + c \]
\( \Phi = 2xy^2 - yz + z + c. \)