§2.2. Field line = Flow line = Streamline = Integral curve.

Suppose \( \mathbf{v}(x,y) \) or \( \mathbf{v}(x,y,z) \) is a stationary velocity field of a fluid. The trajectory of a particle moving along the flow is a flowline. Denote it by \( \mathbf{r}(t) \). It satisfies

\[
\frac{d}{dt} \mathbf{r}(t) = \mathbf{v}(\mathbf{r}(t))
\]

Every flow line = one trajectory of solutions of \( \otimes \)

Flows of §2.1 Ex1: (cf. Math 215/225)

(a)
\[
\begin{align*}
\dot{x} &= x \\
\dot{y} &= y
\end{align*}
\]

(b)
\[
\begin{align*}
\dot{x} &= x \\
\dot{y} &= -y
\end{align*}
\]

(c)
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= x
\end{align*}
\]

(d)
\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= x
\end{align*}
\]
Flow lines of Ex4. pendulum

\[ y = \theta' \]

\[ \theta = 0 \ldots \text{mass at bottom}, \quad \theta = \pm \pi \ldots \text{mass at top} \]

skip:

Eq 2.2.4 – the end of §2.2

See Math 215/255, 345
§ 2.3 Conservative vector fields

**Defn** A vector field \( \mathbf{F} \) is called **conservative** if
\[
\mathbf{F} = \nabla \phi
\]
the gradient of some scalar function \( \phi \) (phi).

We say \( \phi \) is a **potential** of \( \mathbf{F} \).

**Rk** i). If \( \phi \) is a potential of \( \mathbf{F} \), so is
\[
\psi = \phi + C, \quad C = \text{const.}
\]
Hence the choice of \( C \) is not essential.

ii). In physics, opposite sign is used
\[
\mathbf{F} = - \nabla \psi
\]

iii). Conservative means **conservation of energy**

iv). A level set of \( \phi \), \( \phi = C \), is called
an **equipotential** curve in 2D

an **surface** in 3D.
Theorem

The energy

\[ E = \frac{1}{2} m \left( \frac{d\vec{r}}{dt} \right)^2 - \Phi(\vec{r}(t)) \]

for a particle of mass \( m \) and position \( \vec{r}(t) \), velocity \( \vec{v}(t) \) in a conservative force field \( \vec{F} = \nabla \Phi \) is constant in time.

Proof

Let \( \vec{v} = \frac{d\vec{r}}{dt} \). By Newton’s law,

\[ m\vec{v}' = \vec{F} = \nabla \Phi(\vec{r}) \]

Hence

\[ \frac{d}{dt} E(\vec{r}(t)) = m\vec{v}' \cdot \vec{v} - \nabla \Phi(\vec{r}) \cdot \frac{d\vec{r}}{dt} = \nabla \Phi(\vec{r}) \cdot \vec{v} - \nabla \Phi(\vec{r}) \cdot \vec{v} = 0. \]

\[ E(\vec{r}(t)) = E(\vec{r}(0)) = \text{const}. \]

Ex.1

Ex.1 of §2.1

(a) \((x, y) = \nabla \Phi, \Phi = \frac{1}{2}(x^2 + y^2) + C\) \(\begin{cases} \Phi_x = x, \Phi_y = y \\ \Phi = \int x \, dx = \frac{1}{2} x^2 + g(y) \\ \Phi_y = 0 + g'(y) = y \\ g(y) = \frac{1}{2} y^2 + C \end{cases} \)

(b) \((x, -y) = \nabla \Phi, \Phi = \frac{1}{2}(x^2 - y^2)\)

(c) \((y, x) = \nabla \Phi, \Phi = xy\)

(d) \((-y, x) = \nabla \Phi \begin{cases} \Phi_x = -y, \Phi_y = x, \Phi = \int (-y) \, dx = -xy + g(y) \\ \Phi_y = -x + g'(y) = x, \quad g'(y) = 2x \quad \text{impossible} \end{cases} \)

Alternatively, if \begin{cases} \Phi_x = -y, \Phi_y = -x \quad \text{impossible if \( \Phi \) is} \\ \Phi_y = x \quad \text{2nd continuously differentiable} \\ \Phi_{yx} = 1 \end{cases} \]

Conclusion: (a), (b), (c) are conservative, (d) is not.