§ 1.1  curves & their derivatives

A curve is a vector-valued function of 1 variable

\[ \vec{r}(t) = (x(t), y(t)) \text{ or } \vec{r}(t) = (x(t), y(t), z(t)) \]

usually \( t \) can be time and \( \vec{r}(t) \) is the position of a particle at time \( t \). The set

\[ \{ \vec{r}(t) : \ a \leq t \leq b \} \]

is a curve in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) :

\[ \vec{r}(a) \to \vec{r}(b) \]

\( t \) can be a parameter that is not time.

EX 1  The circle \( x^2 + y^2 = a^2 \) is given by

\[ \vec{r}(\theta) = (a \cos \theta, a \sin \theta) \]

\[ 0 \leq \theta \leq 2\pi \]

\[ \vec{r}_2(x) = (x, \sqrt{a^2 - x^2}) \]

\[ -1 \leq x \leq 1 \]

is part of the circle

\[ \vec{r}_3(x) = (x, -\sqrt{a^2 - x^2}), \quad -1 \leq x \leq 1 \]

is the other part.
Exercise \( \vec{F}_1(t) = (a \sin(2t), a \cos(2t)) \),
\( 0 \leq t \leq \pi \)
also describes the circle.

Curves often occur as the intersection of 2 surfaces.

EX2. The intersection of the 2 spheres
\[ S_1: (x-1)^2 + y^2 + z^2 = 1 \]
\[ S_2: x^2 + (y-1)^2 + z^2 = 1 \]
is a circle. The difference of the two equations is
\[-2x + 2y = 0, \quad \text{or} \quad x = y.\]

If we use \( x = y = t \) as parameter, we get
\[ z^2 = 1 - t^2 + 2t - 1 - t^2 = 2t - 2t^2 \]
\[ 2t - 2t^2 \geq 0 \quad \Rightarrow \quad 0 \leq t \leq 1 \]

The fan \( \vec{F}_1(t) = (t, t, \sqrt{2t - 2t^2}) \)
only gives part of the circle similar to Ex 1.

From \( z^2 + 2(t-\frac{1}{2})^2 = 2 \), \( 2z^2 + t(t-\frac{1}{2})^2 = 1 \)
we can parameterize \( z = \sqrt{2} \cos \theta, t = \frac{1}{2} \sin \theta + \frac{1}{2} \)
\( \vec{F}_2(t) = \left( \frac{1}{2} \sin \theta + \frac{1}{2}, \frac{1}{2} \sin \theta + \frac{1}{2}, \sqrt{2} \cos \theta \right) \)
\( 0 \leq \theta \leq 2\pi \).
Viewed from top: \( P \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \) is the center.

radius = \( \frac{\sqrt{2}}{2} \).

\[ |OP| = |PQ| \]