1. (§16.4 #20) If a circle $C$ with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point $P$ on $C$ traces out a curve called an **epicycloid**, with parametric equations

$$x = 5 \cos t - \cos 5t, \quad y = 5 \sin t - \sin 5t.$$ 

Find the area it encloses.

**Solution.** Choose $\vec{F} = (P,Q) = (0,x)$. We have $Q_x - P_y = 1$. (Any other choice with $Q_x - P_y = 1$ will work, but the choice $P = 0$ makes the calculation simple.)

Note the curve $C$ is positively oriented and $0 \leq t < 2\pi$ gives the whole curve. (One way to see it is to note that it is a perturbation of $x = 5 \cos t, y = 5 \sin t$.)

By Green Theorem, the area

$$A = \int_C Qdy = \int_0^{2\pi} x(t)y'(t)dt = \int_0^{2\pi} (5 \cos t - \cos 5t)(5 \cos t - 5 \cos 5t)dt$$

$$= \int_0^{2\pi} (25 \cos^2 t - 30 \cos t \cos 5t + 5 \cos^2 5t) dt$$

Note that

$$\int_0^{2\pi} \cos^2 kt \ dt = \pi, \quad \int_0^{2\pi} \cos kt \cos mt \ dt = 0 \quad (k \neq m),$$

using

$$\cos kt \cos mt = \frac{1}{2} (\cos (k + m)t + \cos (k - m)t).$$

Thus

$$A = 25\pi - 0 + 5\pi = 30\pi.$$
2. (21) (a) If $C$ is the segment connecting the point $(x_1, y_1)$ to the point $(x_2, y_2)$, show that
\[ \int_C xy - ydx = x_1y_2 - x_2y_1. \]
(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is
\[ A = \frac{1}{2} \left[ (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n) \right]. \]
(c) Find the area of the pentagon with vertices $(0, 0), (2, 1), (1, 3), (0, 2),$ and $(-1, 1)$.

**Solution.**
(a) The segment $C$ can be parametrized by
\[ x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1), \quad (0 \leq t \leq 1). \]
Thus
\[ \int_C xy - ydx = \int_0^1 [x(t)y'(t) - y(t)x'(t)] dt \]
\[ = \int_0^1 \left\{ [x_1 + t(x_2 - x_1)](y_2 - y_1) - [y_1 + t(y_2 - y_1)](x_2 - x_1) \right\} dt \]
\[ = \int_0^1 (x_1y_2 - x_2y_1) dt \]
\[ = x_1y_2 - x_2y_1. \]

(b) Denote the line segment from $(x_i, y_i)$ to $(x_{i+1}, y_{i+1})$ as $C_i$, $i = 1, \ldots, n$, with the convention $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. Denote $C = C_1 \cup \cdots \cup C_n$. Since these vertices are oriented counterclockwise, $C$ is positively oriented with respect to the polygon. If we denote $(P, Q) = (-y, x)$, we have $Q_x - P_y = 2$. By Green Theorem and part (a), the area
\[ A = \frac{1}{2} \int_C xy - ydx = \frac{1}{2} \sum_{i=1}^n \int_{C_i} xy - ydx = \frac{1}{2} \sum_{i=1}^n (x_iy_{i+1} - x_{i+1}y_i). \]
(c) Note that we can drop $(0, 2)$ and have the same polygon. By (b), the area
\[ A = \frac{1}{2} \left[ (0) + (6 - 1) + (1 + 3) + (0) \right] = \frac{9}{2}. \]

3. Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$,
\[ \vec{F}_1(x, y) = \frac{-y\hat{i} + (x + 1)\hat{j}}{(x + 1)^2 + y^2}, \quad \vec{F}_2(x, y) = \frac{-y\hat{i} + (x - 1)\hat{j}}{(x - 1)^2 + y^2}, \quad \vec{F}_3(x, y) = y^2\hat{i} + 2xy\hat{j} \]
and $C$ is the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, oriented counterclockwise.

**Solution.** Inside the ellipse, $\vec{F}_1$ has a singularity at $(-1, 0)$, $\vec{F}_2$ has a singularity at $(1, 0)$, and $\vec{F}_3$ has no singularity. Denote $\vec{F}_i = (P_i, Q_i)$ for $i = 1, 2, 3.$
Note that $\vec{F}_1(x, y) = \vec{F}_0(x + 1, y)$ and $\vec{F}_2(x, y) = \vec{F}_0(x - 1, y)$ with $\vec{F}_0(x, y) = (P, Q) = \frac{-y \hat{i} + x \hat{j}}{x^2 + y^2}$. Since we already shown $Q_x - P_y = 0$ for $(x, y) \neq (0, 0)$ in class, we have $\partial_x Q_1 - \partial_y P_1 = 0$ for $(x, y) \neq (-1, 0)$ and $\partial_x Q_2 - \partial_y P_2 = 0$ for $(x, y) \neq (1, 0)$.

Let $C_0$, $C_1$, and $C_2$ denotes the circles $\vec{r}_1(t) = (a_i + \cos t, \sin t), 0 \leq t < 2\pi$, with $a_i = 0, -1, 1$, respectively. By the corollary of the Green Theorem,

$$\int_C \vec{F}_1 \cdot d\vec{r} = \int_{C_1} \vec{F}_1 \cdot d\vec{r} = \int_{C_0} \vec{F}_0 \cdot d\vec{r} = 2\pi.$$  

In the second equality we have used $\vec{F}_1(x, y) = \vec{F}_0(x + 1, y)$. (we could also calculate directly using the above parametrization for $C_1$.) Similarly, $\int_C \vec{F}_2 \cdot d\vec{r} = 2\pi$.

We also have $\int_C \vec{F}_3 \cdot d\vec{r} = \int_D (\partial_x Q_3 - \partial_y P_3) = \int_D (2y - 2y) = 0$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi + 2\pi + 0 = 4\pi.$$

4. (§16.5#1) Find the curl and the divergence of the vector field

$$\vec{F}(x, y, z) = xy^2 z^2 \hat{i} + x^2 yz^2 \hat{j} + x^2 y^2 z \hat{k}.$$  

Solution.

$$\begin{align*}
\text{curl}\vec{F} &= \langle (x^2 y^2 z)_y - (x^2 y z^2)_z, (x y^2 z^2)_z - (x^2 y^2 z)_x, (x^2 y z^2)_x - (x y^2 z^2)_y \rangle \\
&= \langle 2x^2 yz - 2x^2 yz, 2xy^2 z - 2xy^2 z, 2xyz^2 - 2xyz^2 \rangle \\
&= (0, 0, 0).
\end{align*}$$  

$$\text{div}\vec{F} = y^2 z^2 + x^2 z^2 + x^2 y^2.$$

5. (#6) Find the curl and the divergence of the vector field

$$\vec{F}(x, y, z) = \ln(2y + 3z) \hat{i} + \ln(x + 3z) \hat{j} + \ln(x + 2y) \hat{k}.$$  

Solution.

$$\begin{align*}
\text{curl}\vec{F} &= \langle (\ln(x + 2y))_y - (\ln(x + 3z))_z, (\ln(2y + 3z))_z - (\ln(x + 2y))_x, \\
&(\ln(x + 3z))_x - (\ln(2y + 3z))_y \rangle \\
&= \left\langle \frac{2}{x + 2y} - \frac{3}{x + 3z}, \frac{3}{2y + 3z} - \frac{1}{x + 2y}, \frac{1}{x + 3z} - \frac{2}{2y + 3z} \right\rangle \\
\text{div}\vec{F} &= 0 + 0 + 0 = 0.
\end{align*}$$

6. (#9-11) The vector field $\vec{F} = (P, Q, 0)$ is shown in the $xy$-plane and looks the same in all other horizontal planes. That is, $\vec{F}$ is independent of $z$ and its $z$-component is 0.

(a) Is $\text{div}\vec{F}$ positive, negative, or zero? Explain.

(b) Determine whether $\text{curl}\vec{F} = 0$. If not, in which direction does $\text{curl}\vec{F}$ point?
Solution. Note

\[ \text{div} \vec{F} = P_x + Q_y, \quad \text{curl} \vec{F} = (0, 0, Q_x - P_y) \]

For #9, note \( P = 0 \) and \( Q = Q(y) \) with \( Q_y < 0 \). Thus \( \text{div} \vec{F} = Q_y < 0 \), and \( \text{curl} \vec{F} = (0, 0, Q_x - P_y) = 0 \).

For #10, note \( P = P(x) \) with \( P_x > 0 \) and \( Q = Q(y) \) with \( Q_y > 0 \). Thus \( \text{div} \vec{F} = P_x + Q_y > 0 \), and \( \text{curl} \vec{F} = (0, 0, Q_x - P_y) = 0 \).

For #11, note \( P = P(y) \) with \( P_y > 0 \) and \( Q = 0 \). Thus \( \text{div} \vec{F} = P_x + Q_y = 0 \), and \( \text{curl} \vec{F} = (0, 0, Q_x - P_y) \) with \( F_3 < 0 \). Thus \( \text{curl} \vec{F} \) points to negative \( z \)-direction.

7. (#12) Let \( f \) be a scalar function and \( \vec{F} \) a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar function or a vector field.

(a) \( \text{curl} f \), \quad (b) \( \text{grad} f \),
(c) \( \text{div} \vec{F} \), \quad (d) \( \text{curl} (\text{grad} f) \),
(e) \( \text{grad} \vec{F} \), \quad (f) \( \text{grad} (\text{div} \vec{F}) \),
(g) \( \text{div} (\text{grad} f) \), \quad (h) \( \text{grad} (\text{div} f) \),
(i) \( \text{curl} (\text{curl} \vec{F}) \), \quad (j) \( \text{div} (\text{div} \vec{F}) \),
(k) \( (\text{grad} f) \times (\text{div} \vec{F}) \), \quad (l) \( \text{div} (\text{curl}(\text{grad} f)) \)

Solution.

(a) \( \text{curl} f \) is meaningless because \( f \) is a scalar function.
(b) \( \text{grad} f \) is a vector field.
(c) \( \text{div} \vec{F} \) is a scalar function.
(d) \( \text{curl}(\text{grad} f) \) is a vector field.
(e) \( \text{grad} \vec{F} \) is meaningless because \( \vec{F} \) is not a scalar function.
(f) \( \text{grad}(\text{div} \, \vec{F}) \) is a vector field.
(g) \( \text{div}(\text{grad} \, f) \) is a scalar function.
(h) \( \text{grad}(\text{div} \, f) \) is meaningless because \( f \) is a scalar function.
(i) \( \text{curl}(\text{curl} \, \vec{F}) \) is a vector field.
(j) \( \text{div}(\text{curl} \, \vec{F}) \) is meaningless because \( \text{div} \, \vec{F} \) is a scalar function.
(k) \( (\text{grad} \, f) \times (\text{div} \, \vec{F}) \) is meaningless because \( \text{div} \, \vec{F} \) is a scalar function.
(l) \( \text{div}(\text{curl}(\text{grad} \, f)) \) is a scalar function.

8. (#14) Determine whether or not the vector field \( \vec{F} = xyz^4 \hat{i} + x^2z^4 \hat{j} + 4x^2yz^3 \hat{k} \) is conservative. If it is conservative, find a potential function \( f \) such that \( \vec{F} = \nabla f \).

\text{Solution.} \qquad \text{curl} \, \vec{F} = (4x^2z^3 - 4x^2z^3, -8xyz^3 + 4xyz^3, 2xz^4 - xz^4) \neq 0.

Thus \( \vec{F} \) is not conservative.

9. (#17) Determine whether or not the vector field \( \vec{F} = e^{yz} \hat{i} + xze^{yz} \hat{j} + xye^{yz} \hat{k} \) is conservative. If it is conservative, find a potential function \( f \) such that \( \vec{F} = \nabla f \).

\text{Solution.} \qquad \text{curl} \, \vec{F} = (\partial_y(xye^{yz}) - \partial_z(xze^{yz}), \partial_z(e^{yz}) - \partial_x(xye^{yz}), \partial_x(xze^{yz}) - \partial_y(e^{yz})) = 0.

Thus \( \vec{F} \) is conservative. Since \( f_x = F_1 = e^{yz} \), we have \( f(x, y, z) = \int e^{yz} \, dx = xe^{yz} + h(y, z) \).

Since \( f_y = xze^{yz} + h_y = F_2 = xze^{yz} \), we get \( h_y = 0 \) and hence \( h = h(z) \).
Since \( f_z = xye^{yz} + h'(z) = F_3 = xye^{yz} \), we get \( h'(z) = 0 \) and hence \( h = c \) constant.
We conclude \( f(x, y, z) = xe^{yz} + c. \)

10. (#20) Is there a vector field \( \vec{G} \) on \( \mathbb{R}^3 \) such that \( \text{curl} \, \vec{G} = (x, y, z) \)? Explain.

\text{Solution.} \quad \text{No, otherwise} \quad 0 = \text{div} \, \text{curl} \, \vec{G} = \text{div}(x, y, z) = 1 + 1 + 1 = 3.

11. (#31) Let \( \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \) and \( r = |\vec{r}| \). Verify the following identities:

(a) \( \nabla r = \vec{r}/r \), \hspace{1cm} (b) \( \nabla \times \vec{r} = 0 \),
(c) \( \nabla(1/r) = -\vec{r}/r^3 \), \hspace{1cm} (d) \( \nabla \ln r = \vec{r}/r^2 \).
Solution.  

(a)  
\[ \nabla r = (\partial_x r, \partial_y r, \partial_z r) = \frac{\vec{r}}{r} \]

(b)  
\[ \nabla \times \vec{r} = (\partial_y z - \partial_z y, \partial_z x - \partial_x z, \partial_x y - \partial_y x) = 0 \]

(c) By Chain rule and (a),  
\[ \nabla \left( \frac{1}{r} \right) = -r^{-2} \nabla r = -\frac{\vec{r}}{r^3} \]

(d) By Chain rule and (a),  
\[ \nabla \ln r = r^{-1} \nabla r = \frac{\vec{r}}{r^2} \]