1. (§16.2#15) Evaluate the line integral \( I = \int_C z^2 dx + x^2 dy + y^2 dz \), where \( C \) is the line segment from \((1, 0, 0)\) to \((4, 1, 2)\).

**Solution.** Parametrize \( C \) by
\[
\mathbf{r}(t) = (1, 0, 0) + t(3, 1, 2) = (1 + 3t, t, 2t), \quad 0 \leq t \leq 1.
\]
We have \( \mathbf{r}'(t) = (3, 1, 2) \), and
\[
I = \int_0^1 (2t^2 + 3t^2 + 1) dt = \frac{35}{3}.
\]

2. (#22) Evaluate the line integral \( I = \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F} = x \mathbf{i} + y \mathbf{j} + xy \mathbf{k} \) and \( C \) is given by the vector function \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}, 0 \leq t \leq \pi \).

**Solution.** We have \( \mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}, \) and
\[
I = \int_0^\pi \cos t \sin t dt = \frac{1}{2} \int_0^\pi \sin(2t) dt = \frac{1}{2} \left[ -\frac{1}{4} \cos 2t \right]_0^\pi = 0.
\]

3. (#36) Find the mass and center of mass of a wire in the shape of the helix \( x = t, \ y = \cos t, \ z = \sin t, \) \( 0 \leq t \leq 2\pi \), if the density at any point is equal to the square of the distance from the origin.

Hint. \( \int t^2 \cos t dt = (t^2 - 2) \sin t + 2t \cos t, \) \( \int t^2 \sin t dt = (-t^2 + 2) \cos t + 2t \sin t \)

**Solution.** Note
\[
\mathbf{r}'(t) = (1, -\sin t, \cos t), \quad |\mathbf{r}'(t)| = \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2}.
\]
The density is \( \sigma = x^2 + y^2 + z^2 \). Thus
\[
\sigma(\mathbf{r}(t)) = t^2 + \cos^2 t + \sin^2 t = t^2 + 1.
\]
The mass is
\[
m = \int_C \sigma ds = \int_0^{2\pi} (t^2 + 1) \sqrt{2} dt = \sqrt{2} \left[ \frac{1}{3} t^3 + t \right]_0^{2\pi} = \sqrt{2} \left[ \frac{8}{3} \pi^3 + 2\pi \right].
\]
The center of mass is (you can also compute it component-wisely)
\[
\mathbf{r} = \frac{1}{m} \int_C \sigma \mathbf{r} ds = \frac{1}{m} \int_0^{2\pi} (t^2 + 1)(t, \cos t, \sin t) dt
\]
\[
= \frac{\sqrt{2}}{m} \int_0^{2\pi} (t^3 + t, (t^2 + 1) \cos t, (t^2 + 1) \sin t) dt
\]
\[
= \frac{\sqrt{2}}{m} \left[ \left( \frac{t^4}{4} + \frac{t^2}{2}, (t^2 - 1) \sin t + 2t \cos t, (-t^2 + 1) \cos t + 2t \sin t \right) \right]_0^{2\pi}
\]
\[
= \frac{1}{\sqrt{2}} \left[ \left( \frac{t^4}{4} + \frac{t^2}{2}, (t^2 - 1) \sin t + 2t \cos t, (-t^2 + 1) \cos t + 2t \sin t \right) \right]_0^{2\pi}
\]
\[
= \frac{3}{\sqrt{2} \pi^3 + 2} (\pi, \pi, -\pi).
\]

4. (#39) Find the work done by the force field \( \mathbf{F}(x, y) = x \hat{i} + (y + 2) \hat{j} \) in moving an object along an arch of the cycloid
\[
\mathbf{r}(t) = (t - \sin t) \hat{i} + (1 - \cos t) \hat{j}, \quad 0 \leq t \leq 2\pi.
\]

**Solution.** We have \( \mathbf{F}'(t) = (1 - \cos t, \sin t), \) and
\[
I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} ((t - \sin t)(1 - \cos t) + (1 - \cos t + 2 \sin t) \sin t) dt
\]
\[
= \int_0^{2\pi} (t - t \cos t + 2 \sin t) dt
\]
\[
= \left[ t^2/2 - t \sin t - \cos t - 2 \cos t \right]_0^{2\pi} = 2\pi^2.
\]

5. (#41) Find the work done by the force field \( \mathbf{F}(x, y, z) = (x - y^2, y - z^2, z - x^2) \) on a particle that moves along the line segment from \((0, 0, 1)\) to \((2, 1, 0)\).

**Solution.** Parametrize the line segment by
\[
\mathbf{r}(t) = (0, 0, 1) + t[(2, 1, 0) - (0, 0, 1)] = (2t, t, 1 - t), \quad 0 \leq t \leq 1.
\]
We have \( \mathbf{r}'(t) = (2, 1, -1), \) and
\[
I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t - t^2, t - (1 - t)^2, 1 - t - 4t^2) \cdot (2, 1, -1) dt
\]
\[
= \int_0^1 (t^2 + 8t - 2) dt
\]
\[
= \left[ \frac{t^3}{3} + 4t^2 - 2t \right]_0^1 = 7/3.
\]

6. (#51) An object moves along the curve \( C \) shown in the figure (Figure 1) from \((1, 2)\) to \((9, 8)\). The lengths of the vectors in the force field \( \mathbf{F} \) are measured in newtons by the scales on the axes. Estimate the work done by \( \mathbf{F} \) on the object.
Solution. Let us denote the line segment from (1, 2) to (9, 2) by $C_1$, and the line segment from (9, 2) to (9, 8) by $C_2$. On $C_1$, we can use $x$ as the parameter. We have

$$\vec{F}(x) = (2, g(x)), \quad \vec{r}(x) = (x, 2), \quad 1 \leq x \leq 9,$$

for some function $g(x)$. Thus

$$I_1 = \int_{C_1} \vec{F} \cdot d\vec{r} = \int_1^9 (2, g(x)) \cdot (1, 0) \, dx = \int_1^9 2 \, dx = 16.$$ 

On $C_2$, we can use $y$ as the parameter. We have

$$\vec{F}(y) = (h(y), 1), \quad \vec{r}(y) = (9, y), \quad 2 \leq y \leq 8,$$

for some function $h(y)$. Thus

$$I_2 = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_2^8 (h(y), 1) \cdot (0, 1) \, dy = \int_2^8 dy = 6.$$ 

The work is

$$\int_C \vec{F} \cdot d\vec{r} = I_1 + I_2 = 16 + 6 = 22.$$

7. (§16.3#1) The figure (Figure 2) shows a curve $C$ and a contour map of a function $f$ whose gradient is continuous. Find $\int_C \nabla f \cdot d\vec{r}$.

Solution. Denote the initial and terminal points of $C$ by $P$ and $Q$. We have

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P) = 50 - 10 = 40.$$
8. (#2) A table of values of a function $f$ with continuous gradient is given in Figure 3. Find $\int_C \nabla f \cdot d\mathbf{r}$ where $C$ has parametric equations

$$x = t^2 + 1, \quad y = t^3 + t, \quad 0 \leq t \leq 1.$$ 

**Solution.**

We have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x(1), y(1)) - f(x(0), y(0)) = f(2, 2) - f(1, 0) = 9 - 3 = 6.$$

9. (#11) The figure (Figure 4) shows the vector field $\mathbf{F}(x, y) = (2xy, x^2)$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.

(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.

(b) What is this common value?

**Solution.**

(a) By inspection, $\mathbf{F} = \nabla f$ with $f = x^2y$. Since $\mathbf{F}$ is conservative, the line integral is independent of the path.

(b) 

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 2) - f(1, 2) = 18 - 2 = 16.$$

The original Problems 10-12 are postponed to H5 and hence removed. Do not hand them in with H4 since they will not be marked!