1. (§13.4#16) Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position:

\[ \vec{a}(t) = \sin t \hat{i} + 2 \cos t \hat{j} + 6t \hat{k}, \quad \vec{v}(0) = -\hat{k}, \quad \vec{r}(0) = \hat{j} - 4\hat{k}. \]

**Solution.** We have

\[
\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)du \\
= (0, 0, -1) + \int_0^t (\sin u, 2 \cos u, 6u)du \\
= (1 - \cos t, 2 \sin t, 3t^2 - 1)
\]

\[
\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)du \\
= (0, 1, -4) + \int_0^t (1 - \cos u, 2 \sin u, 3u^2 - 1)du \\
= (0, 1, -4) + (t - \sin t, 2 - 2 \cos t, t^3 - t) \\
= (t - \sin t, 3 - 2 \cos t, t^3 - t - 4)
\]

2. (#23) A projectile is fired with an initial speed of 200 m/s and angle of elevation 60°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

**Solution.** Let \( \vec{r}(t) \) denote the position of the projectile, \( \vec{v} = \vec{r}' \), and \( \vec{a} = \vec{v}' \). We have

\[
\vec{r}(0) = (0, 0), \quad \vec{v}(0) = (200 \cos 60^\circ, 200 \sin 60^\circ) = (100, 100\sqrt{3}), \quad \vec{a} = (0, -g).
\]

We have

\[
\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)du = (100, 100\sqrt{3} - gt).
\]

\[
\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)du = (100t, 100\sqrt{3}t - \frac{1}{2}gt^2).
\]

Let \( t_1 > 0 \) denote the time the projectile lands. We have \( y(t_1) = 100\sqrt{3}t_1 - \frac{1}{2}gt_1^2 = 0 \), thus

\[
t_1 = \frac{200\sqrt{3}}{g}.
\]

The range is

\[
x(t_1) = 100t_1 = \frac{20000\sqrt{3}}{g}
\]
The maximum height is

\[ y(t_1/2) = 50\sqrt{3}t_1 - \frac{1}{8}gt_1^2 = \frac{15000}{g} \]

Since \( \vec{v}(t_1) = (100, -100\sqrt{3}) \), the speed at impact is

\[ |\vec{v}(t_1)| = 100\sqrt{1 + 3} = 200. \]

If we take \( g = 9.8\, \text{m/s}^2 \), then \( x(t_1) \approx 3534.8 \) and \( y(t_1/2) \approx 1530.6 \).

3. (#28) A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle 50° above the horizontal. Is it a home run?

Solution. The gravity constant is \( g = 32.174 \, \text{ft/s}^2 \). Let \( \vec{r}(t) \) denote the position of the ball, \( \vec{v} = \vec{r}' \), and \( \vec{a} = \vec{v}' \). We have

\[ \vec{r}(0) = (0, 3), \quad \vec{v}(0) = (115\cos 50°, 115\sin 50°) \quad \vec{a} = (0, -g). \]

We have

\[ \vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)du = \vec{v}(0) + (0, -gt). \]

\[ \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)du = (115(\cos 50°)t, 3 + 115(\sin 50°)t - \frac{1}{2}gt^2). \]

Suppose the ball flies over the fence at time \( t_1 > 0 \). We have

\[ x(t_1) = 400 = 115(\cos 50°)t_1, \quad t_1 = \frac{400}{115\cos 50°} = 5.411. \]

At time \( t_1 \) the height of the ball is at least 10 for a home run, that is

\[ 10 \leq y(t_1) = 3 + 115(\sin 50°)t_1 - \frac{1}{2}gt_1^2. \]

We get

\[ g \leq 32.0822, \]

which is incorrect. Hence it is not a home run.

Remark. If you take \( g = 32 \), which is less accurate, then the answer becomes YES.

4. (#32) A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?

Solution. Let the east be the positive \( x \) direction, the north be the positive \( y \) direction, and the ball be initially at the origin. Let \( \vec{r}(t) = (x, y, z)(t) \) denote the position of the ball, \( \vec{v} = \vec{r}' \), and \( \vec{a} = \vec{v}' \). We have

\[ \vec{r}(0) = (0, 0, 0), \quad \vec{v}(0) = (0, -30\cos 30°, 30\sin 30°) \quad \vec{a} = (4/0.8, 0, -g). \]

We have

\[ \vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)du = (5t, -15\sqrt{3}, 15 - gt). \]
\[ \mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{\ddot{r}}(u) \, du = \left( \frac{5}{2} t^2, -15\sqrt{3} t, 15 t - \frac{1}{2} g t^2 \right). \]

Suppose the ball lands at time \( t_1 > 0 \). We have \( 0 = z(t_1) = 15 t_1 - \frac{1}{2} g t_1^2 \), thus

\[ t_1 = \frac{30}{g} \approx 15.306 \text{ (s)}, \]

if we take \( g = 9.8 \text{m/s}^2 \). The ball lands at

\[ \mathbf{r}(t_1) = \left( \frac{2250}{g^2}, \frac{-450\sqrt{3}}{g}, 0 \right) \approx (23.43, -79.53, 0), \]

with landing velocity \( \mathbf{\ddot{r}}(t_1) = \left( \frac{150}{g}, -15\sqrt{3}, -15 \right) \) and speed

\[ |\mathbf{\ddot{r}}(t_1)| = \left| \left( \frac{150}{g}, -15\sqrt{3}, -15 \right) \right| \approx 33.68 \text{ (m/s)} \]

5. (§16.1) Sketch the vector field \( \mathbf{F} = x \mathbf{i} + (x+y) \mathbf{j} \).

**Solution.**

\[ \mathbf{F} = (x, x+y) \]

6. (#24) Find the gradient vector field \( \nabla f \) of \( f(x,y,z) = x^2 ye^{y/z} \).

**Solution.**

\[ \nabla f = (2xye^{y/z}, x^2(1+y/z)e^{y/z}, -x^2 y^2 e^{y/z} z^2) \]

7. (#25) Find the gradient vector field \( \nabla f \) of \( f(x,y) = \frac{1}{2} (x-y)^2 \) and sketch it.

**Solution.** \( \nabla f = (x-y, y-x) \).
8. (§16.2#2) Evaluate the line integral \( I = \int_C (x/y) \, ds \), where \( C \) is the curve \( x = t^3, \ y = t^4, \ 1 \leq t \leq 2 \).

**Solution.**

\[
(x', y') = (3t^2, 4t^3), \quad |(x', y')| = (9t^4 + 16t^6)^{1/2} = t^2(9 + 16t^2)^{1/2}
\]

\[
I = \int_1^2 t^2(9 + 16t^2)^{1/2} \, dt = \int_1^2 t(9 + 16t^2)^{1/2} \, dt
\]

Let \( u = 9 + 16t^2 \) so that \( du = 32t \, dt \), we get

\[
I = \int_{25}^{73} \frac{1}{32} u^{1/2} \, du = \frac{1}{48} \left[u^{3/2}\right]_{25}^{73} = \frac{1}{48} (73^{3/2} - 125)
\]

9. (#3) Evaluate the line integral \( I = \int_C xy^4 \, ds \), where \( C \) is the right half of the circle \( x^2 + y^2 = 16 \).

**Solution.** Parametrize the right half circle by

\[
\vec{r}(t) = (4 \cos t, 4 \sin t), \quad -\frac{\pi}{2} \leq t < \frac{\pi}{2}.
\]

We have

\[
\vec{r}'(t) = (-4 \sin t, 4 \cos t), \quad |\vec{r}'(t)| = 4.
\]

Thus

\[
I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 \cos t)(4 \sin t)^4 \, dt
\]

Let \( u = \sin t \), so that \( du = \cos t \, dt \), and

\[
I = 4^6 \int_{-1}^{1} u^4 \, du = 4^6 \left[ \frac{1}{5} u^5 \right]_{-1}^{1} = \frac{2^{13}}{5} = 1638.4
\]
10. (#5) Evaluate the line integral \( I = \int_C (x^2y + \sin x) \, dy \), where \( C \) is the arc of the parabola \( y = x^2 \) from \((0,0)\) to \((\pi, \pi^2)\).

**Solution.** Parametrize the arc of the parabola by

\[ \mathbf{r}(t) = (t, t^2), \quad 0 \leq t < \pi. \]

We have \( y' = 2t \), and

\[
I = \int_0^\pi (t^2t^2 + \sin t)2t \, dt = \int_0^\pi (2t^5 + 2t \sin t) \, dt
\]

\[
= \left[ \frac{t^6}{3} - 2t \cos t + 2 \sin t \right]_0^\pi = \frac{\pi^6}{3} + 2\pi.
\]

11. (#7) Evaluate the line integral \( I = \int_C (x + 2y) \, dx + x^2 \, dy \), where \( C \) consists of line segments from \((0,0)\) to \((2,1)\) and from \((2,1)\) to \((3,0)\).

**Solution.** Parametrize the two line segments by

\[ C_1 : \mathbf{r}(t) = (2t, t), \quad 0 \leq t \leq 1 \]

and

\[ C_2 : \mathbf{q}(t) = (2 + t, 1 - t), \quad 0 \leq t \leq 1. \]

We have \( \mathbf{r}'(t) = (2, 1) \) and \( \mathbf{q}'(t) = (1, -1) \), and

\[
I = \int_0^1 ((2t + 2t)2 + (2t)^21) \, dt + \int_0^1 ((2 + t + 2(1 - t))1 + (2 + t)^2(-1)) \, dt
\]

\[
= \int_0^1 (3t^2 + 3t) \, dt = \left[ t^3 + \frac{3}{2}t^2 \right]_0^1 = \frac{5}{2}.
\]

12. (#13) Evaluate the line integral \( I = \int_C xy e^{yz} \, dy \) where \( C \) is the curve \( x = t, y = t^2, z = t^3, 0 \leq t \leq 1. \)

**Solution.** We have \( y' = 2t \) and

\[
I = \int_0^1 t \cdot t^2 e^{t^5}2t \, dt = \int_0^1 2t^4 e^{t^5} \, dt
\]

Let \( u = t^5 \). Then \( du = 5t^4 \, dt \) and

\[
I = \frac{2}{5} \int_0^1 e^u \, du = \frac{2}{5}(e - 1)
\]