1. Let $s$ be the arclength function of a curve $\vec{r}$. Show that $\left|\frac{d\vec{r}}{ds}\right| = 1$.

   **Solution.** By chain rule and inverse function rule,
   
   $$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{r}}{dt} \cdot \left(\frac{ds}{dt}\right)^{-1},$$
   
   $$\left|\frac{d\vec{r}}{ds}\right| = \left|\frac{d\vec{r}}{dt}\right| \cdot \left|\frac{ds}{dt}\right|^{-1}.$$

   However, by the definition of arclength,
   
   $$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|.$$

   Thus $\left|\frac{d\vec{r}}{ds}\right| = 1$.

   **Remark 1.** This identity gives another way to understand that $s$ is the arclength.

   **Remark 2.** Note that $\frac{d\vec{r}}{ds} = T$.

2. (#4) Find the length of the curve

   $$\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + \ln(\cos t) \hat{k}, \quad 0 \leq t \leq \pi/4.$$

   **Solution.**

   $$\vec{r}'(t) = (\sin t, \cos t, -\sin t).$$

   $$\left|\vec{r}'(t)\right| = \left(\sin^2 t + \cos t^2 + \frac{\sin^2 t}{\cos^2 t}\right)^{1/2} = \left(\frac{1}{\cos^2 t}\right)^{1/2} = \frac{1}{\cos t}.$$ 

   We have $\left|\vec{r}'(t)\right| = \frac{1}{\cos t}$ for $0 \leq t \leq \pi/4$. Thus

   $$L = \int_0^{\pi/4} \left|\vec{r}'(u)\right|du = \int_0^{\pi/4} \frac{du}{\cos u} = \left[\ln |\sec u + \tan u|\right]_0^{\pi/4}$$

   $$= \ln |\sqrt{2} + 1| - \ln |1 + 0| = \ln |\sqrt{2} + 1|.$$ 

3. (#6) Find the length of the curve

   $$\vec{r}(t) = t^2 \hat{i} + 9t \hat{j} + 4t^{3/2} \hat{k}, \quad 1 \leq t \leq 4.$$

   **Solution.**

   $$\vec{r}'(t) = (2t, 9, 6t^{1/2}).$$

   $$\left|\vec{r}'(t)\right| = (4t^2 + 81 + 36t)^{1/2} = |2t + 9| = 2t + 9,$$

   for $1 \leq t \leq 4$. Thus

   $$L = \int_1^4 \left|\vec{r}'(u)\right|du = \int_1^4 (2u + 9)du = \left[u^2 + 9u\right]_1^4 = 16 + 36 - 10 = 42.$$
4. (#11) Let $C$ be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of $C$ from the origin to the point $(6, 18, 36)$.

**Solution.** If we let $x = t$, we can solve $y = \frac{1}{2}t^2$ and $z = \frac{1}{3}xy = \frac{1}{6}t^3$. Let $\mathbf{r}(t) = (x, y, z)(t)$. The origin is $\mathbf{r}(0)$ while $(6, 18, 36) = \mathbf{r}(6)$. Thus the curve is for $0 \leq t \leq 6$. We have

\[
\mathbf{r}'(t) = (1, t, \frac{1}{2}t^2),
\]

\[
|\mathbf{r}'(t)| = \left(1 + t^2 + \frac{1}{4}t^4\right)^{1/2} = 1 + \frac{1}{2}t^2.
\]

Thus

\[
L = \int_{0}^{6} |\mathbf{r}'(t)|dt = \int_{0}^{6} \left(1 + \frac{1}{2}t^2\right)dt = \left[t + \frac{1}{6}t^3\right]_{0}^{6} = 42.
\]

5. (#14) (a) Find the arc length function for the curve

\[
\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + \sqrt{2} e^t \mathbf{k}
\]

measured from the point $P(0, 1, \sqrt{2})$ in the direction of increasing $t$, and then reparametrize the curve respect to arc length starting from $P$.

(b) Find the point 4 units along the curve in the direction of increasing $t$ from $P$.

**Solution.** (a) We have

\[
\mathbf{r}'(t) = (e^t \sin t + cos t, e^t(- \sin t + \cos t), \sqrt{2} e^t),
\]

\[
|\mathbf{r}'(t)| = (e^{2t} \left((\sin t + \cos t)^2 + (- \sin t + \cos t)^2 + 2\right))^{1/2} = 2e^t.
\]

Thus

\[
s(t) = \int_{0}^{t} |\mathbf{r}'(u)|du = \int_{0}^{t} 2e^u du = 2e^t - 2.
\]

We can solve

\[
e^t = \frac{s + 2}{2}, \quad t = \ln \frac{s + 2}{2},
\]

and get

\[
\mathbf{r} = \frac{s + 2}{2} \left(\sin \left(\ln \frac{s + 2}{2}\right) \mathbf{i} + \cos \left(\ln \frac{s + 2}{2}\right) \mathbf{j} + \mathbf{k}\right).
\]

(b) The point is

\[
\mathbf{r}|_{s=4} = 3 \sin (\ln 3) \mathbf{i} + 3 \cos (\ln 3) \mathbf{j} + 3 \mathbf{k}.
\]

6. (#19) Find the unit tangent vector $T(t)$, unit normal vector $N(t)$, and the curvature $\kappa(t)$ of the curve

\[
\mathbf{r}(t) = (\sqrt{2}t, e^t, e^{-t}).
\]

**Solution.** We have

\[
\mathbf{r}'(t) = (\sqrt{2}, e^t, -e^{-t}),
\]
\[ |\vec{r}'(t)| = (2 + e^{2t} + e^{-2t})^{1/2} = e^t + e^{-t}. \]

Thus
\[ T(t) = \frac{\vec{r}'}{|\vec{r}'|} = \frac{(\sqrt{2}, e^t, -e^{-t})}{e^t + e^{-t}}. \]

We have
\[ T'(t) = \frac{(0, e^t, -e^{-t})(e^t + e^{-t}) - (\sqrt{2}, e^t, -e^{-t})(e^t - e^{-t})}{(e^t + e^{-t})^2} = \frac{(-\sqrt{2}(e^t - e^{-t}), 2, 2)}{(e^t + e^{-t})^2} \]
\[ |T'(t)| = \frac{\sqrt{2}}{e^t + e^{-t}}. \]

Thus
\[ N(t) = \frac{T'(t)}{|T'(t)|} = \frac{(-e^t - e^{-t}, \sqrt{2}, \sqrt{2})}{e^t + e^{-t}}. \]

The curvature
\[ \kappa(t) = \frac{|T'(t)|}{|\vec{r}'(t)|} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}. \]

7. (#20) Find the unit tangent vector $T(t)$, unit normal vector $N(t)$, and the curvature $\kappa(t)$ of the curve
\[ \vec{r}(t) = (t, \frac{1}{2} t^2, t^2). \]

Solution. We have
\[ \vec{r}'(t) = (1, t, 2t), \]
\[ |\vec{r}'(t)| = (1 + t^2 + 4t^2)^{1/2} = (1 + 5t^2)^{1/2}. \]

Thus
\[ T(t) = \frac{\vec{r}'}{|\vec{r}'|} = \frac{(1, t, 2t)}{(1 + 5t^2)^{1/2}}. \]

We have
\[ T'(t) = \frac{(0, 1, 2)}{(1 + 5t^2)^{1/2}} - \frac{(1, t, 2t)}{2(1 + 5t^2)^{3/2}} \cdot 10t = \frac{(-5t, 1, 2)}{(1 + 5t^2)^{3/2}} \]
\[ |T'(t)| = \frac{\sqrt{5}}{1 + 5t^2}. \]

Thus
\[ N(t) = \frac{T'(t)}{|T'(t)|} = \frac{(-5t, 1, 2)}{\sqrt{5}(1 + 5t^2)^{1/2}}. \]

The curvature
\[ \kappa(t) = \frac{|T'(t)|}{|\vec{r}'(t)|} = \frac{\sqrt{5}}{(1 + 5t^2)^{3/2}}. \]
8. (#24) Find the curvature of \( \vec{r}(t) = (t^2, \ln t, t \ln t) \) at the point \((1, 0, 0)\).

Solution. The point \((1, 0, 0) = \vec{r}(1)\). We have

\[
\vec{r}'(t) = (2t, t^{-1}, 1 + \ln t), \quad \vec{r}'(1) = (2, 1, 1),
\]

\[
\vec{r}''(t) = (2, -t^{-2}, t^{-1}), \quad \vec{r}''(1) = (2, -1, 1)
\]

We have \((2, 1, 1) \times (2, -1, 1) = (2, 0, -4)\). The curvature

\[
\kappa |_{t=1} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} |_{t=1} = \frac{|(2, 0, -4)|}{|2, 1, 1|^3} = \frac{\sqrt{20}}{6^{3/2}} = \frac{\sqrt{30}}{18}.
\]

9. (#31) At what point does the curve \( y = e^x \) have maximum curvature? What happens to the curvature as \( x \to \infty \)?

Solution. For \( f(x) = e^x \), we have \( f'(x) = e^x \). The curvature

\[
\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}
\]

We have \( \kappa(x) > 0 \) for all \( x \) and \( \kappa(x) \to 0 \) as \( |x| \to \infty \). Thus max \( \kappa \) occurs at an interior point where \( dk/dx = 0 \). We have

\[
\frac{dk}{dx} = \frac{e^x}{(1 + e^{2x})^{3/2}} - \frac{3e^x}{2(1 + e^{2x})^{5/2}} \cdot 2e^{2x} = \frac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}
\]

Thus max \( \kappa \) occurs when \( e^{2x} = 1/2 \), i.e., \( x = \frac{1}{2} \ln \frac{1}{2} = -\frac{1}{2} \ln 2 \).

10. (#49) Find equations of the normal plane and osculating plane of the curve

\[
x = \sin 2t, \quad y = -\cos 2t, \quad z = 4t,
\]

at the point \( P(0, 1, 2\pi) \).

Solution. Let \( \vec{r}(t) = (x, y, z)(t) = (\sin 2t, -\cos 2t, 4t) \). Since \( z = 4t = 2\pi \), the point \( P = \vec{r}(\frac{\pi}{2}) \). We have

\[
\vec{r}'(t) = (2 \cos 2t, 2 \sin 2t, 4), \quad |\vec{r}'(t)| = 2\sqrt{5},
\]

\[
T(t) = \frac{\vec{r}'}{|\vec{r}'|} = \frac{1}{\sqrt{5}}(\cos 2t, \sin 2t, 2)
\]

\[
T'(t) = \frac{1}{\sqrt{5}}(-2 \sin 2t, 2 \cos 2t, 0), \quad |T'(t)| = \frac{2}{\sqrt{5}}.
\]

Thus

\[
N(t) = \frac{T'(t)}{|T'(t)|} = (-\sin 2t, \cos 2t, 0).
\]

At the point \( P(0, 1, 2\pi) = \vec{r}(\frac{\pi}{2}) \),

\[
T = \frac{1}{\sqrt{5}}(-1, 0, 2), \quad N = (0, -1, 0), \quad B = T \times N = \frac{1}{\sqrt{5}}(2, 0, 1).
\]
The normal plane passes $P$ and is orthogonal to $T$, thus has the equation

$$-x + 2(z - 2\pi) = 0.$$ 

The osculating plane passes $P$ and is orthogonal to $B$, thus has the equation

$$2x + (z - 2\pi) = 0.$$ 

11. (#51) Find equations of the osculating circles of the ellipse $9x^2 + 4y^2 = 36$ at the points $(2,0)$ and $(0,3)$. Graph the ellipse and the two osculating circles.

**Solution.** Near $(2,0)$, the ellipse is given by

$$x = g(y) = (4 - \frac{4}{9}y^2)^{1/2}, \quad 2 = g(0).$$

We have

$$g'(y) = -\frac{4}{9}y(4 - \frac{4}{9}y^2)^{-1/2}, \quad g''(y) = -\frac{4}{9}(4 - \frac{4}{9}y^2)^{-1/2} - \frac{16}{81}y^2(4 - \frac{4}{9}y^2)^{-3/2}$$

$$g'(0) = 0, \quad g''(0) = -\frac{2}{9}$$

$$\kappa = \frac{|g''(0)|}{[1 + (g'(0))^2]^{1/2}} = \frac{2}{9}, \quad r = \frac{1}{\kappa} = \frac{9}{2}.$$

The normal is $N = (-1,0)$, the center is $(2,0) + rN = (-\frac{5}{2},0)$, and the osculating circle is

$$(x + \frac{5}{2})^2 + y^2 = \frac{81}{4}.$$ 

Near $(0,3)$, the ellipse is given by

$$y = f(x) = (9 - \frac{9}{4}x^2)^{1/2}, \quad 3 = f(0).$$

We have

$$f'(x) = -\frac{9}{4}x(9 - \frac{9}{4}x^2)^{-1/2}, \quad f''(x) = -\frac{9}{4}(9 - \frac{9}{4}x^2)^{-1/2} - \frac{81}{16}x^2(9 - \frac{9}{4}x^2)^{-3/2}$$

$$f'(0) = 0, \quad f''(0) = -\frac{3}{4}$$

$$\kappa = \frac{|f''(0)|}{[1 + (f'(0))^2]^{1/2}} = \frac{3}{4}, \quad r = \frac{1}{\kappa} = \frac{4}{3}.$$ 

The normal is $N = (0,-1)$, the center is $(0,3) + rN = (0, \frac{5}{3})$, and the osculating circle is

$$x^2 + (y - \frac{5}{3})^2 = \frac{16}{9}.$$
Alternative solution. One can parametrize the ellipse by 
\[ \vec{r}(t) = (2 \cos t, 3 \sin t) \]
with \((2, 0) = \vec{r}(0)\) and \((0, 3) = \vec{r}(\frac{\pi}{2})\). We have 
\[ \vec{r}'(t) = (-2 \sin t, 3 \cos t), \quad \vec{r}''(t) = (-2 \cos t, -3 \sin t) \]
\[ \vec{r}'(t) \times \vec{r}''(t) = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} \]
\[ \kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{6}{(4 \sin^2 t + 9 \cos^2 t)^{3/2}}. \]
At \((2, 0) = \vec{r}(0)\), \(\kappa = \frac{6}{9^{3/2}} = \frac{2}{9}\). At \((0, 3) = \vec{r}(\pi/2)\), \(\kappa = \frac{6}{(4)^{3/2}} = \frac{3}{2}\). One then find the equations of the osculating circles in the same way as the previous solution.

12. (#55) Find equations of the normal plane and osculating plane of the curve of intersection of the parabolic cylinders \(x = y^2\) and \(z = x^2\) at the point \(P(1, 1, 1)\).

Solution. If we let \(y = t\), we can solve \(x = t^2\) and \(z = t^4\). Let 
\[ \vec{r}(t) = (x, y, z)(t) = (t^2, t, t^4). \]
The point \(P\) is \(\vec{r}(1)\). We have 
\[ \vec{r}'(t) = (2t, 1, 4t^3), \]
\[ |\vec{r}'(t)| = (4t^2 + 1 + 16t^6)^{1/2}. \]
Thus 
\[ T(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = (4t^2 + 1 + 16t^6)^{-1/2} (2t, 1, 4t^3), \]
\[ T'(t) = (4t^2 + 1 + 16t^6)^{-1/2} (2, 0, 12t^2) - \frac{1}{2} (4t^2 + 1 + 16t^6)^{-3/2} (8t + 96t^5)(2t, 1, 4t^3). \]

At \( t = 1 \), we have

\[ T = \frac{1}{\sqrt{21}} (2, 1, 4), \quad T' = \frac{1}{\sqrt{21}} (2, 0, 12) - \frac{52}{(21)^{3/2}} (2, 1, 4) = \frac{2(-31, -26, 22)}{(21)^{3/2}}. \]

Thus \( N \) is parallel to \( N_1 = (-31, -26, 22) \), and \( B \) is parallel to

\[ B_1 = (2, 1, 4) \times (-31, -26, 22) = (126, -168, -21) \]

The normal plane passes \( P \) and is orthogonal to \( T \), thus has the equation

\[ 2(x - 1) + 1(y - 1) + 4(z - 1) = 0. \]

The osculating plane passes \( P \) and is orthogonal to \( B_1 \), thus has the equation

\[ 126(x - 1) - 168(y - 1) - 21(z - 1) = 0, \]

or

\[ 6(x - 1) - 8(y - 1) - (z - 1) = 0. \]