MATH 317 Assignment 9

§4.2, §4.3

1. Redo §3.4 Example 5 using Divergence theorem:
   Compute the outward flux of \( \mathbf{F}(x, y, z) = (-2x, y, z) \) across the boundary \( S \) of the burrito solid enclosed by \( y^2 + z^2 = 9, x = 0 \) and \( x + y = 5 \).

2. Repeat the previous problem for \( \mathbf{F}(x, y, z) = (3x, 2y, z) \).

3. Compute the outward flux \( \int_S \mathbf{F} \cdot d\mathbf{S} \) where \( \mathbf{F}(x, y, z) = y^2 \cos z \mathbf{i} + z^2 \cos x \mathbf{j} + 2xz \mathbf{k} \) and \( S \) is the boundary of the tetrahedron with vertices \((0,0,0), (1,0,0), (0,2,0), \) and \((0,0,3)\).

4. Compute the outward flux \( \int_S \mathbf{F} \cdot d\mathbf{S} \) where \( \mathbf{F}(x, y, z) = \frac{2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \) and \( S \) is the boundary of the solid \( V \) enclosed by \( z = x^2 + y^2 - 10 \) and \( z = \cos(x + y) + 10 \).

5. (pipe flow) Suppose water flows in the pipe \( P : y^2 + z^2 \leq a^2, a > 0 \), with velocity
   \[ \mathbf{F}(x, y, z) = (g(y, z), 0, 0) \]
   where the function \( g(y, z) \) is smooth and independent of \( x \). Let \( S_1 \) and \( S_2 \) be any two disjoint cross sections of the pipe, given by \( x = f_1(y, z) \) and \( x = f_2(y, z) \) for \((y, z) \in D = \{(y, z) : y^2 + z^2 \leq a^2\}\), with normal vectors in the \( x \)-increasing direction. Show that
   \[ \iint S_1 \mathbf{F} \cdot d\mathbf{S} = \iint S_2 \mathbf{F} \cdot d\mathbf{S}. \]

   Remark. The amount of water per unit time through any cross section is the same.

6. Compute the line integral \( I = \oint_C (x - 2y)dx + (3x - 4y)dy \) for the polygon \( C \) connecting \((1,1), (0,2), (-1,1), (-1,-1), (0,0), (1,-1)\) and then back to \((1,1)\).
   Check the orientation of the curve before applying the Green’s theorem.

7. Evaluate the line integral \( I = \oint_C (x^9 - x^4y - \frac{2}{3}x^2y^3)dx + (y^9 + xy^4)dy \) where \( C \) is the positively oriented circle \( x^2 + y^2 = 9 \).

8. Let \( \mathbf{F}(x, y) = \frac{y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}, \quad \mathbf{G}(x, y) = \mathbf{F}(x - 5, y) + \mathbf{F}(x + 5, y) \).
   Calculate \( I = \oint_C \mathbf{G} \cdot d\mathbf{r} \) where
   (a) \( C \) is the circle \((x - 2)^2 + y^2 = 1, \) positively oriented.
   (b) \( C \) is the circle \((x - 5)^2 + y^2 = 1, \) positively oriented.
   (c) \( C \) is the circle \((x - 5)^2 + y^2 = 16, \) positively oriented.
   (d) \( C \) is the circle \((x - 2)^2 + y^2 = 16, \) positively oriented.
   (e) \( C \) is the circle \((x - 2)^2 + y^2 = 100, \) positively oriented.
9. If a circle $C_1$ with radius 1 rolls along the outside of the circle $x^2 + y^2 = 9$, a fixed point $P$ on $C_1$ traces out a curve called an **epicycloid**, with parametric equations

$$x = 4 \cos t - \cos 4t, \quad y = 4 \sin t - \sin 4t, \quad 0 \leq t < 2\pi.$$  

Find the area it encloses.