MATH 317  Assignment 2

§1.3–§1.6, §2.1

1. The function \( f(x) = x^3 - x \) has local minimum \( f\left(\frac{1}{\sqrt{3}}\right) = \frac{-2\sqrt{3}}{9} \). Find the centres of the osculating circles of the curve \( y = x^3 - x \) at the points \( P\left(\frac{1}{\sqrt{3}}, -\frac{2\sqrt{3}}{9}\right) \) and \( Q(1, 0) \).

   \textbf{Solution.} We have \( f'(x) = 3x^2 - 1 \) and \( f''(x) = 6 \). The curvature is
   \[
   \kappa(x) = \frac{|f''|}{(1 + (f')^2)^{3/2}}
   \]
   Thus
   \[
   \kappa\left(\frac{1}{\sqrt{3}}\right) = \frac{\frac{6\sqrt{3}}{9}}{(1 + 0)^{3/2}} = \frac{2\sqrt{3}}{3}, \quad \kappa(1) = \frac{6}{(1 + 2)^{3/2}} = \frac{6}{5^{3/2}}.
   \]
   At \( P \), as a local minimum, \( \mathbf{N} = (0, 1) \), and
   \[
   \mathbf{r}_c = P + \frac{1}{\kappa} \mathbf{N} = \left(\frac{1}{\sqrt{3}} - \frac{2\sqrt{3}}{9}, \frac{\sqrt{3}}{6}\right) + (0, 1) = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{18}\right)
   \]
   At \( Q \), as \( f'(1) = 2 \), a tangent vector is \( (1, 2) \). The normal vector \( \mathbf{N} \) is orthogonal to \( (1, 2) \) and pointing upward as the curve is concave up. Thus
   \[
   \mathbf{N} = \frac{1}{\sqrt{5}}(-2, 1),
   \]
   and
   \[
   \mathbf{r}_c = Q + \frac{1}{\kappa} \mathbf{N} = (1, 0) + \frac{5^{3/2}}{6} \cdot \frac{1}{\sqrt{5}}(-2, 1) = (\frac{-2}{3}, \frac{5}{6})
   \]

2. Find the unit tangent, unit normal and binormal vectors and the curvature and torsion of the curve
   \[
   \mathbf{r}(t) = \sin(3t) \, \mathbf{i} + \cos(3t) \, \mathbf{j} + 4t \, \mathbf{k}.
   \]

   \textbf{Solution.}
   \[
   \mathbf{r}'(t) = 3\cos(3t) \, \mathbf{i} - 3\sin(3t) \, \mathbf{j} + 4 \, \mathbf{k}.
   \]
   \[
   |\mathbf{r}'(t)| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 16} = 5.
   \]
   \[
   \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{3\cos(3t) \, \mathbf{i} - 3\sin(3t) \, \mathbf{j} + 4 \, \mathbf{k}}{5}.\]
   \[
   \mathbf{T}'(t) = -\frac{9\sin(3t) \, \mathbf{i} - 9\cos(3t) \, \mathbf{j}}{5}, \quad |\mathbf{T}'| = \frac{9}{5}
   \]
   \[
   \kappa(t) = \frac{|\mathbf{T}'|}{|\mathbf{T}'|} = \frac{9/5}{9} = \frac{9}{25}.
   \]
   \[
   \mathbf{N}(t) = \frac{\mathbf{T}'}{|\mathbf{T}'|} = -\sin(3t) \, \mathbf{i} - \cos(3t) \, \mathbf{j}.
   \]
4. Suppose that the curve $C$ is the intersection of the cylinder $x^2 + y^2 = 1$ with the surface $z = x^2 - y^2$.

(a) Find a parameterization of $C$.

(b) Determine the curvature of $C$ at the point $P = (1/\sqrt{2}, 1/\sqrt{2}, 0)$.

(c) Find the osculating plane to $C$ at the point $P$. In general, the osculating plane to a curve $\mathbf{r}(t)$ at the point $\mathbf{r}(t_0)$ is the plane which fits the curve best at $\mathbf{r}(t_0)$. It passes through $\mathbf{r}(t_0)$ and has normal vector $\mathbf{B}(t_0)$.

(d) Find the radius and the centre of the osculating circle to $C$ at the point $P$.

**Solution.** (a) The curve $x^2 + y^2 = 1$ is a circle of radius 1. So we can parametrize it by $x(\theta) = \cos \theta$, $y(\theta) = \sin \theta$, $0 \leq \theta < 2\pi$. The $z$-coordinate of any point on the intersection is determined by $z = x^2 - y^2$. So we can use the parametrization

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + [\cos^2 \theta - \sin^2 \theta] \mathbf{k}$$

$$= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \cos(2\theta) \mathbf{k}, \quad 0 \leq \theta < 2\pi.$$
(b) Note that \( \vec{r}(\theta) = P \) when \( \theta = \pi/4 \). For general \( \theta \), the velocity and acceleration are

\[
\vec{v}(\theta) = \vec{r}'(\theta) = -\sin \theta \hat{i} + \cos \theta \hat{j} - 2\sin(2\theta) \hat{k}
\]
\[
\vec{a}(\theta) = \vec{v}'(\theta) = -\cos \theta \hat{i} - \sin \theta \hat{j} - 4\cos(2\theta) \hat{k}
\]

In particular,

\[
\vec{v}(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} - 2 \hat{k}
\]
\[
\vec{a}(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j}
\]
\[
|\vec{v}(\frac{\pi}{4})| = \sqrt{5}
\]
\[
\vec{v}(\frac{\pi}{4}) \times \vec{a}(\frac{\pi}{4}) = -\sqrt{2} \hat{i} + \sqrt{2} \hat{j} + \hat{k}
\]
\[
|\vec{v}(\frac{\pi}{4}) \times \vec{a}(\frac{\pi}{4})| = \sqrt{5}
\]

So the curvature

\[
\kappa(\frac{\pi}{4}) = \frac{|\vec{v}(\frac{\pi}{4}) \times \vec{a}(\frac{\pi}{4})|}{|\vec{v}(\frac{\pi}{4})|^3} = \frac{1}{5}.
\]

(c) The binormal to \( C \) at \( P \) is

\[
\hat{B} = \frac{\vec{v}(\frac{\pi}{4}) \times \vec{a}(\frac{\pi}{4})}{|\vec{v}(\frac{\pi}{4}) \times \vec{a}(\frac{\pi}{4})|} = \frac{-\sqrt{2} \hat{i} + \sqrt{2} \hat{j} + \hat{k}}{\sqrt{5}}
\]

So the osculating plane to \( C \) at \( P \) is

\[
(-\sqrt{2}, \sqrt{2}, 1) \cdot (x - \frac{1}{\sqrt{2}}, y - \frac{1}{\sqrt{2}}, z) = 0,
\]

or

\[
z = \sqrt{2}x - \sqrt{2}y.
\]

(d) From the computations in parts (b) and (c), and \( \hat{T} = \frac{\vec{v}}{|\vec{v}|} \), we have

\[
\hat{N}(\frac{\pi}{4}) = \hat{B}(\frac{\pi}{4}) \times \hat{T}(\frac{\pi}{4}) = \frac{-\sqrt{2} \hat{i} + \sqrt{2} \hat{j} + \hat{k}}{\sqrt{5}} \times \frac{-\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} - 2 \hat{k}}{\sqrt{5}} = \frac{-\hat{i} - \hat{j}}{\sqrt{2}}
\]

So the osculating circle has radius \( 1/\kappa(\frac{\pi}{4}) = 5 \) and centre

\[
\vec{r}_c(\frac{\pi}{4}) = \vec{r}(\frac{\pi}{4}) + \frac{1}{\kappa(\frac{\pi}{4})} \hat{N}(\frac{\pi}{4}) = (1/\sqrt{2}, 1/\sqrt{2}, 0) - 5(1/\sqrt{2}, 1/\sqrt{2}, 0) = (-2\sqrt{2}, -2\sqrt{2}, 0)
\]

5. Find the mass and centre of mass of the curve

\[
\vec{r}(t) = t^2 \hat{i} + 2t \hat{j} + \frac{1}{3} t^3 \hat{k}, \quad 0 \leq t \leq 2,
\]

with density \( \rho(t) = t^2 \).

\textit{Solution}. By Problem 3,

\[
|\vec{r}'(t)| = t^2 + 2.
\]
Thus the mass is
\[ M = \int \rho \, ds = \int_0^2 t^2(t^2 + 2) \, dt = \left[ \frac{1}{5} t^5 + \frac{2}{3} t^3 \right]_0^2 = \frac{32}{5} + \frac{16}{3} = \frac{176}{15} \]

The centre of mass \((\bar{x}, \bar{y}, \bar{z})\) is given by
\[
\bar{x} = \frac{1}{M} \int x \rho \, ds = \frac{1}{M} \int_0^2 t^2 t^2(t^2 + 2) \, dt = \frac{1}{M} \left[ \frac{1}{7} t^7 + \frac{2}{5} t^5 \right]_0^2 = \frac{15}{176} \left( \frac{128}{7} + \frac{64}{5} \right) = \frac{204}{77} 
\]
\[
\bar{y} = \frac{1}{M} \int y \rho \, ds = \frac{1}{M} \int_0^2 2t^2(t^2 + 2) \, dt = \frac{1}{M} \left[ \frac{1}{3} t^6 + t^4 \right]_0^2 = \frac{15}{176} \left( \frac{64}{3} + 16 \right) = \frac{35}{11} 
\]
\[
\bar{z} = \frac{1}{M} \int z \rho \, ds = \frac{1}{M} \int_0^2 \frac{1}{3} t^3 t^2(t^2 + 2) \, dt = \frac{1}{3M} \left[ \frac{1}{8} t^8 + \frac{1}{3} t^6 \right]_0^2 = \frac{5}{176} \left( 32 + \frac{64}{3} \right) = \frac{50}{33} 
\]

6. Suppose Mr. Hinton hit a baseball 1 m above the ground toward the centre field fence, which is 3 m higher than and 120 m from the home plate. Suppose the ball leaves the bat with 40 m/s speed at angle 50° above the horizontal. Is it a home run?

Hint. \( \cos 50° \approx 0.6428, \sin 50° \approx 0.7660, \) gravity constant \( g = 9.8 \text{ m/s}^2. \)

**Solution.** Denote the trajectory of the ball as
\[ \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j}. \]

We have
\[ \vec{r}(0) = \hat{j}, \quad \vec{r}'(0) = 40 \cos 50° \hat{i} + 40 \sin 50° \hat{j}, \quad \vec{r}''(t) = -g \hat{j}. \]

Let \( t_1 \) be the time such that the ball reaches the fence, \( x(t_1) = 120. \) The question is whether or not \( y(t_1) > 3? \) We have
\[
\vec{r}'(t) = \vec{r}'(0) + \int_0^t \vec{r}'' \, dt = 40 \cos 50° \hat{i} + (40 \sin 50° - gt) \hat{j} 
\]
\[
\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}' \, dt = 40t \cos 50° \hat{i} + (1 + 40t \sin 50° - \frac{1}{2}gt^2) \hat{j} 
\]

Thus \( 40t_1 \cos 50° = 120, \) \( t_1 = \frac{3}{\cos 50°}, \)
\[
y(t_1) - 3 = (1 + 40t_1 \sin 50° - \frac{1}{2}gt_1^2) - 3 = -2 + 120 \frac{\sin 50°}{\cos 50°} - 4.9 \left( \frac{3}{\cos 50°} \right)^2 
\]
\[
= -2 + 143.01 - 106.73 > 0. 
\]

Yes, it is a home run!

7. Sketch each of the following vector fields, by drawing a figure like Figure 2.1.1 in the CLP-IV text.

(a) \( \vec{v}(x, y) = 2x \hat{i} - \hat{j}. \)  
(b) \( \vec{v}(x, y) = \frac{y \hat{i} - x \hat{j}}{\sqrt{x^2 + y^2}}. \)

**Solution.** (a) The vertical component of \( \vec{v}(x, y) = 2x \hat{i} - \hat{j} \) is always \(-1. \) Its horizontal component is \( 2x, \) so that
* \( \vec{v}(x, y) \) is rightward pointing when \( x > 0 \) and leftward pointing when \( x < 0, \) and
* the magnitude of the horizontal component grows linearly with the distance from the \( y \)-axis.
It is sketched in the figure on the left below.

\[ \mathbf{v}(x, y) = \mathbf{y} \mathbf{i} - x \mathbf{j} / \sqrt{x^2 + y^2} \]

* is of length 1 and
* is perpendicular to the radius vector \( x \mathbf{i} + y \mathbf{j} \).
* \( \mathbf{v}(x, y) \) is rightward pointing when \( y > 0 \) and leftward pointing when \( y < 0 \), and
* \( \mathbf{v}(x, y) \) is downward pointing when \( x > 0 \) and upward pointing when \( x < 0 \).

It is sketched in the figure on the right above.

8. Let \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} \) be the two dimensional vector field sketched below. Determine the signs of \( P, Q, \frac{\partial Q}{\partial x} \) and \( \frac{\partial Q}{\partial y} \) at the point \( A \).

Solution. The arrows near the point \( A \) are pointing to the right, indicating that \( P > 0 \), and upward, indicating that \( Q > 0 \). Moving from left to right near \( A \), the vertical component of the arrows is decreasing, indicating that \( \frac{\partial Q}{\partial x} < 0 \). Moving vertically upwards near \( A \), the vertical component of the arrows is increasing, indicating that \( \frac{\partial Q}{\partial y} > 0 \).