MATHEMATICS 317 April 2001 Final Exam

[10] 1) Find and sketch the field lines of the vector field \( \vec{F} = x \hat{i} + 3y \hat{j} \).

[15] 1) Let \( C \) be the curve from \( (0, 0, 0) \) to \( (1, 1, 1) \) along the intersection of the surfaces \( y = x^2 \) and \( z = x^3 \).
   a) Find \( \int_C \vec{F} \cdot d\vec{r} \) if \( \vec{F} = (xz - y) \hat{i} + (x + z) \hat{j} + y \hat{k} \).
   b) Find \( \int_C \rho \, ds \) if \( s \) is arclength along \( C \) and \( \rho = 8x + 36z \).
   c) Find \( \int_C \vec{F} \cdot d\vec{r} \) if \( \vec{F} = \sin y \hat{i} + (x \cos y + z) \hat{j} + (y + z) \hat{k} \).

[15] 3) Let \( S \) be the portion of the elliptical cylinder \( x^2 + \frac{1}{4}y^2 = 1 \) lying between the planes \( z = 0 \) and \( z = 1 \) and let \( \hat{n} \) denote the outward normal to \( S \). Let \( \vec{F} = x \hat{i} + xyz \hat{j} + zy^3 \hat{k} \). Calculate the flux integral \( \int_S \vec{F} \cdot \hat{n} \, dS \) directly, using an appropriate parameterization of \( S \).

[15] 4) Let \( S \) be the portion of the sphere \( x^2 + y^2 + (z - 1)^2 = 4 \) that lies above the \( xy \)-plane. Find the flux of \( \vec{F} = (x^2 + e^{y^2}) \hat{i} + (e^{x^2} + y^2) \hat{j} + (4 + 5x) \hat{k} \) outward across \( S \).

[15] 5) Let \( C_1 \) be the circle \( (x - 2)^2 + y^2 = 1 \) and let \( C_2 \) be the circle \( (x - 2)^2 + y^2 = 9 \). Let \( \vec{F} = -\frac{x}{x^2+y^2} \hat{i} + \frac{y}{x^2+y^2} \hat{j} \). Find the integrals \( \oint_{C_1} \vec{F} \cdot d\vec{r} \) and \( \oint_{C_2} \vec{F} \cdot d\vec{r} \).

[15] 6) Let \( C \) be the intersection of the paraboloid \( z = 4 - x^2 - y^2 \) with the cylinder \( x^2 + (y - 1)^2 = 1 \), oriented counterclockwise when viewed from high on the \( z \)-axis. Let \( \vec{F} = xz \hat{i} + x \hat{j} + yz \hat{k} \). Find \( \oint_C \vec{F} \cdot d\vec{r} \).

[15] 7) The following statements may be true or false. Decide which. If true, give a proof. If false, provide a counter-example.
   a) If \( \vec{F} \) is any smooth vector field defined in \( \mathbb{R}^3 \) and if \( S \) is any sphere, then \( \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} \, dS = 0 \). Here \( \hat{n} \) is the outward normal to \( S \).
   b) If \( \vec{F} \) and \( \vec{G} \) are smooth vector fields in \( \mathbb{R}^3 \) and if \( \oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{G} \cdot d\vec{r} \) for every circle \( C \), then \( \vec{F} = \vec{G} \).
   c) Let \( \vec{F} \) and \( \vec{G} \) be smooth vector fields defined in \( \mathbb{R}^3 \). Suppose that, for every circle \( C \), we have \( \oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{G} \cdot \hat{n} \, dS \), where \( S \) is the oriented disk with boundary \( C \). Then \( \vec{G} = \vec{\nabla} \times \vec{F} \).