Be sure this exam has 6 pages including the cover
The University of British Columbia
MATH 317, Section 201, Instructor Tai-Peng Tsai
Midterm Exam II – March 15, 2013

Family Name ____________________________ Given Name ____________________________

Student Number _________________________ Signature _____________________________

No calculators, books, notebooks or any other written materials are allowed

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Rules for the exam:

1. You should sit in odd numbered seats from the nearest aisle. (A seat next to an aisle is number 1.) No two persons are allowed to sit next to each other.

2. From five minutes before the end of the exam, you cannot hand in your exam any more and should wait in your seat until the end of the exam.

3. When the invigilator says that the exam is over, you should stop writing and immediately pass your exam to the end of your row. The invigilator will collect the exams in the aisles.

4. You are not allowed to leave until the invigilator has collected all exams and says that you can leave.

5. Any conversation between students during the exam and the exam collection time is considered cheating. Writing after the end of exam is considered cheating.
1. The vector field \( \vec{F}(x, y, z) = Ax^3 y^2 z \hat{i} + (z^3 + Bx^4 yz) \hat{j} + (3yz^2 - x^4 y^2) \hat{k} \) is conservative on \( \mathbb{R}^3 \).

(a) Find the values of the constants \( A \) and \( B \).

\( \text{Answer.} \) Denote \( F = (P, Q, R) \). Since it is conservative, we have

\[
0 = P_z - R_x = Ax^3 y^2 - (-4x^3 y^2)
\]

and

\[
0 = Q_z - R_y = 3z^2 + Bx^4 y - (3z^2 - 2x^4 y).
\]

Thus

\[
A = -4, \quad B = -2.
\]

\( \text{Answer #2:} \) One may solve the potential function \( \phi \) in part (b) directly and find \( A \) and \( B \) along the way.

(b) Find a potential function \( \phi \) such that \( \vec{F} = \nabla \phi \) on \( \mathbb{R}^3 \) and \( \phi(0, 0, 0) = 2 \).

\( \text{Answer.} \) Since \( \phi_x = P \), we get

\[
\phi = \int P(x, y, z) dx = -x^4 y^2 z + f(y, z).
\]

We have \( \phi_y = -2x^4 yz + f_y(y, z) = Q = z^3 - 2x^4 yz \), thus

\[
f_y(y, z) = z^3, \quad f(y) = \int z^3 dy = yz^3 + g(z).
\]

We have \( \phi_z = -x^4 y^2 + 3yz^2 + g'(z) = R = 3yz^2 - x^4 y^2 \), thus

\[
g'(z) = 0, \quad g(z) = C
\]

for some constant \( C \) and \( \phi(x, y) = -x^4 y^2 z + yz^3 + C \). Since \( \phi(0, 0, 0) = 2 \), we have \( C = 2 \) and

\[
\phi(x, y) = -x^4 y^2 z + yz^3 + 2.
\]

(c) If \( C \) is the curve \( y = -x, z = x^2 \) from \( (0, 0, 0) \) to \( (1, -1, 1) \), evaluate \( I = \int_C \vec{F} \cdot d\vec{r} \).

\( \text{Answer.} \) Since \( \vec{F} \) is conservative,

\[
I = \phi(1, -1, 1) - \phi(0, 0, 0) = (-1 - 1 + 2) - 2 = -2.
\]

\( \text{Answer #2:} \) Let \( \vec{r}(t) = (t, -t, t^2) \) with \( 0 \leq t \leq 1 \). We have \( \vec{r}'(t) = (1, -1, 2t) \) and

\[
I = \int_0^1 (-4t^7, t^6 + 2t^7, -3t^5 - t^6) \cdot (1, -1, 2t) \, dt = \int_0^1(-8t^7 - 7t^6) \, dt = -2.
\]
2. (10 pt) Let \( \mathbf{F}(x,y) = (x^4 - 8x^3 y + 2xy^3 + y^4)\mathbf{i} + (3x - 2x^4 + 3x^2 y^2 + 4xy^3 + y^4)\mathbf{j} \).

(a) Compute \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \) where \( C_1 \) is the straight line segment from \((-1,-1)\) to \((1,1)\).

Answer. Denote \( \mathbf{F} = (P,Q) \) and parametrize \( C_1 \):
\[
\mathbf{r}(t) = (t,t), \quad -1 \leq t \leq 1,
\]
thus \( \mathbf{r}'(t) = (1,1) \) and
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} (-4t^4, 3t + 6t^4) \cdot (1,1) dt = \int_{-1}^{1} 3t + 2t^4 dt = 0 + \left[ \frac{2}{5} t^5 \right]_{-1}^{1} = \frac{4}{5}.
\]

Answer #2: Parametrize \( C_1 \) by \( \mathbf{r}(u) = (2u - 1, 2u - 1), \) \( 0 \leq u \leq 1. \) One has
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2 \int_{0}^{1} 3(2u - 1) + 2(2u - 1)^4 du = \frac{4}{5}.
\]

(b) Compute \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \) where \( C_2 \) is the arc from \((-1,-1)\) to \((1,1)\) along the lower-right half circle \( x^2 + y^2 = 2, \) \( y \leq x. \)

Answer. Let \( D \) be the half disk \( x^2 + y^2 \leq 2 \) and \( y \leq x. \) It is enclosed by \(-C_1\) and \( C_2. \) The negative sign in front of \( C_1 \) indicates a negative orientation. By Green’s theorem,
\[
- \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = I := \iint_D (Q_x - P_y) \, dx \, dy.
\]
Since
\[
Q_x - P_y = (3 - 8x^3 + 6xy^2 + 4y^3 - 0) - (0 - 8x^3 + 6xy^2 + 4y^3) = 3,
\]
we have
\[
I = \iint_D 3 \, dx \, dy = 3 \cdot \text{area}(D) = 3 \cdot \frac{1}{2} \pi (\sqrt{2})^2 = 3\pi.
\]
By (a), we have
\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = I + \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 3\pi + \frac{4}{5}.
\]
(5 pt) 3. (a) Let \( \vec{F}(x, y, z) = (xe^y, \sin(z) \cos(x), x^2 + y^2 + z^2) \). Compute \( \text{curl} \, \vec{F} \) and \( \text{div} \, \vec{F} \).

**Answer.**

\[
\text{curl} \, \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xe^y & \sin(z) \cos(x) & x^2 + y^2 + z^2 \end{vmatrix} = \left( \frac{\partial_y (x^2 + y^2 + z^2) - \partial_z (\sin(z) \cos(x)), \partial_z (xe^y) - \partial_x (x^2 + y^2 + z^2), \partial_x (\sin(z) \cos(x)) - \partial_y (xe^y)}{x^2 + y^2 + z^2} \right) = (2y - \cos(z) \cos(x), -2x, -\sin(z) \sin(x) - xe^y).
\]

\[
\text{div} \, \vec{F} = \partial_x (xe^y) + \partial_y (\sin(z) \cos(x)) + \partial_z (x^2 + y^2 + z^2) = e^y + 2z.
\]

(5 pt) (b) Find the mass of a wire in the shape of the helix \( x = t, y = \cos t, z = \sin t, \) if its density is \( \rho(x, y, z) = x^2 + y^2 + z^2 \).

**Answer.** Denote the helix by \( C \) and the vector function by \( \vec{r}(t) \). We have

\[
\vec{r}'(t) = (1, -\sin t, \cos t), \quad |\vec{r}'| = \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2}.
\]

Hence

\[
m = \int_C \rho ds = \int_0^{2\pi} \rho(\vec{r}(t))|\vec{r}'(t)|dt = \int_0^{2\pi} (t^2 + 1)\sqrt{2} dt = \sqrt{2} \left[ \frac{1}{3}t^3 + t \right]_0^{2\pi} = \frac{8\pi^3 \sqrt{2}}{3} + 2\pi \sqrt{2}.
\]
4. (a) A surface $S$ has the parametric equation $\mathbf{r}(u, v) = (u^2, u - v^2, v^2)$ with $0 \leq u \leq \sqrt{3}$ and $1 \leq v \leq 3$. Find its tangent plane at the point $P(1, -3, 4)$.

**Answer.** At $(1, -3, 4)$, $u = 1, v = 2$.

We have

$$\mathbf{r}_u(P) = (2u, 1, 0) \big|_{P} = (2, 1, 0), \quad \mathbf{r}_v(P) = (0, -2v, 2v) \big|_{P} = (0, -4, 4).$$

Hence

$$\mathbf{r}_u \times \mathbf{r}_v = 4(1, -2, -2).$$

We can choose $(1, -2, -2)$ as a normal vector to the tangent plane at $P$. It has equation

$$x - 2y - 2z = 1(1) - 2(-3) - 2(4) = -1.$$

**Answer #2:** $\mathbf{r}_u \times \mathbf{r}_v = (2u, 1, 0) \times (0, -2v, 2v) = (2v, -4uv, -4uv)$, which is $(4, -8, -8)$ when $u = 1$ and $v = 2$.

(b) Set up the integral, but do not evaluate it, for the area of the part of the paraboloid $x^2 + z^2 + y = 3$ that satisfies $y \geq 2x$.

**Answer.** It is a graph $y = f(x, z) = 3 - x^2 - z^2$ with domain (which is the projection of the surface on $xz$-plane)

$$D : 3 - x^2 - z^2 \geq 2x, \quad i.e., \quad (x + 1)^2 + z^2 \leq 4.$$

The area is

$$\iint_D \sqrt{1 + f_x^2 + f_z^2} \, dz \, dx = \iint_D \sqrt{1 + 4x^2 + 4z^2} \, dx \, dz.$$

One can further replace $\int_D$ by $\int_{x=-3}^{1} \int_{z=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$, or $\int_{x=-3}^{1} \int_{z=-\sqrt{3-x^2}}^{\sqrt{3-x^2}}$, or parametrize $D$ by $x = r \cos \theta - 1, z = r \sin \theta, 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$.

**Remark.** The boundary of the surface is an ellipse which lies on the paraboloid, the plane, and also the cylinder $(x + 1)^2 + z^2 = 4$.

**Answer #2:** The surface is symmetric in $z$. The part $z \geq 0$ is given by $z = g(x, y) = (3 - x^2 - y)^{1/2}$ with domain $D$ on $xy$-plane given by

$$D : -3 \leq x \leq 1, \quad 2x \leq y \leq 3 - x^2.$$

The area (including the part $z < 0$) is

$$2 \int_D \sqrt{1 + g_x^2 + g_y^2} \, dx \, dy = 2 \int_{x=-3}^{1} \int_{y=2x}^{3-x^2} \sqrt{\frac{3.25 - y}{3 - y - x^2}} \, dy \, dx.$$
Formulas for MT2

1. For a curve $\mathbf{r}(t)$, arclength $s = \int_0^t |\mathbf{r}'(\tau)| d\tau$, \( \frac{ds}{dt} = |\mathbf{r}'| \), \( ds = |\mathbf{r}'(t)| dt \)

2. $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$, $\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

3. $\kappa = \left| \frac{dT}{ds} \right| = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$, $\kappa \mathbf{N} = \frac{d\mathbf{T}}{ds}$

4. For $y = f(x)$, $\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$

5. Green’s theorem: $\int_C P dx + Q dy = \iint_D (Q_x - P_y) dA$

6. For a surface $S$ given by $\mathbf{r}(u,v) : D \rightarrow \mathbb{R}^3$, the surface area is $\int_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dudv$

7. For a graph $S$ given by $z = f(x,y)$, $(x,y) \in D$, the surface area is $\int_S dS = \iint_D \sqrt{1 + f_x^2 + f_y^2} dxdy$

8. For a surface of revolution $S$ given by $r = f(z)$, $a \leq z \leq b$, the surface area is $\int_S dS = \int_a^b 2\pi f(z) \sqrt{1 + [f'(z)]^2} dz$