For a given curve $\mathbf{r}(t)$,

1. $s = \int_0^t |\mathbf{r}'(\tau)|d\tau$, \quad $\frac{ds}{dt} = |\mathbf{r}'|$, \quad $ds = |\mathbf{r}'(t)|dt$

2. $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$, \quad $\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$, \quad $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

3. $\kappa = \frac{d\mathbf{T}}{ds} \Bigg|_{\mathbf{T}'} \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$, \quad $\kappa \mathbf{N} = \frac{d\mathbf{T}}{ds}$

4. For $y = f(x)$, \quad $\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$
1. (a) Find a vector function \( \mathbf{r}(t) \) that represents the curve of intersection of the cone \( x^2 + y^2 = z^2 \) and the plane \( z = y + 1 \).

   **Answer.** Substitute the second equation into the first
   \[
   x^2 + y^2 = (y + 1)^2 = y^2 + 2y + 1
   \]
   Thus
   \[
   y = (x^2 - 1)/2
   \]
   If we choose the parameter \( x = t \), then \( y = \frac{1}{2}(t^2 - 1) \), and \( z = \frac{1}{2}(t^2 + 1) \). Thus
   \[
   \mathbf{r}(t) = \left( t, \frac{1}{2}(t^2 - 1), \frac{1}{2}(t^2 + 1) \right), \quad (-\infty < t < \infty).
   \]

   **Remark.** Many of you wrote
   \[
   \mathbf{r}(t) = \left( \sqrt{2t + 1}, t, t + 1 \right)
   \]
   but it only gives the portion of the curve with \( x \geq 0 \).

   (b) Find the length of the curve
   \[
   \mathbf{r}(t) = (\ln t) \hat{i} + 2t \hat{j} + t^2 \hat{k}
   \]
   from \((0, 2, 1)\) to \((1, 2e, e^2)\).

   **Answer.**
   \[
   \mathbf{r}'(t) = \left( t^{-1}, 2, 2t \right), \quad |\mathbf{r}'(t)| = (t^{-2} + 4 + 4t^2)^{1/2} = t^{-1} + 2t
   \]
   The two points correspond to \( t = 1 \) and \( t = e \).
   The arclength is
   \[
   \int_{1}^{e} (t^{-1} + 2t) dt = [\ln t + t^2]_{1}^{e} = e^2.
   \]
(8 points) 2. Find the vectors $T$, $N$ and $B$ of the curve $r(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$ at the point $(1, \frac{2}{3}, 1)$.

Answer. This is Section 13.3 #47.

The point $(1, \frac{2}{3}, 1)$ corresponds to $t = 1$.

$$r'(t) = \langle 2t, 2t^2, 1 \rangle,$$

$$|r'(t)| = (4t^2 + 4t^4 + 1)^{1/2} = 2t^2 + 1$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{2t^2 + 1} \langle 2t, 2t^2, 1 \rangle$$

$$T(1) = \frac{1}{3} \langle 2, 2, 1 \rangle.$$

To compute $N$,

$$T'(t) = \frac{-4t}{(2t^2 + 1)^2} \langle 2t, 2t^2, 1 \rangle + \frac{1}{2t^2 + 1} \langle 2, 4t, 0 \rangle$$

$$T'(1) = -\frac{4}{9} \langle 2, 2, 1 \rangle + \frac{1}{3} \langle 2, 4, 0 \rangle = \frac{1}{9} \langle -2, 4, -4 \rangle$$

$$N(1) = \frac{T'(1)}{|T'(1)|} = \frac{1}{3} \langle -1, 2, -2 \rangle,$$

To compute $B$,

$$B(1) = T(1) \times N(1) = \frac{1}{3} \langle -2, 1, 2 \rangle.$$

Alternative solution #2:

Note $B = \frac{r' \times r''}{|r' \times r''|}$. Since $r''(t) = \langle 2, 4t, 0 \rangle$, $r''(1) = \langle 2, 4, 0 \rangle$, and $r'(1) \times r''(1) = \langle -4, 2, 4 \rangle$, we have

$$B(1) = \frac{S}{|S|} = \frac{1}{3} \langle -2, 1, 2 \rangle.$$

We also have

$$N(1) = T(1) \times B(1) = \frac{1}{3} \langle 2, 2, 1 \rangle \times \frac{1}{3} \langle -2, 1, 2 \rangle = \frac{1}{3} \langle -1, 2, -2 \rangle.$$

Alternative solution #3:

Note $N = \frac{Q}{|Q|}$ where $Q$ is $r''$ minus its projection on $T$ direction. Since $r''(1) = \langle 2, 4, 0 \rangle$,

$$Q = r''(1) - [r''(1) \cdot T(1)]T(1) = (2, 4, 0) - \frac{4}{3} (2, 2, 1) = \frac{2}{3} (-1, 2, -2),$$

$|Q| = 1$ and hence $N(1) = Q$. 
3. (7 points) Find the equation of the osculating circle of the parabola $x = y^2$ at the point $(1, 1)$.

Answer. We consider $x$ as a function $y$. Note

$$\frac{dx}{dy} = 2y, \quad \frac{dx^2}{d^2y} = 2$$

At $(1, 1)$,

$$\frac{dx}{dy} = 2, \quad \frac{dx^2}{d^2y} = 2$$

A tangent vector is $(2, 1)$. A unit normal vector is

$$N = \frac{1}{\sqrt{5}}(1, -2).$$

We do not choose $\frac{1}{\sqrt{5}}(-1, 2)$ because the direction of $N$ is where the curve bends to.

The curvature and radius are

$$\kappa = \frac{2}{(1 + 2^2)^{3/2}} = \frac{2}{5^{3/2}}, \quad R = \frac{1}{\kappa} = \frac{5^{3/2}}{2}.$$

The center is

$$(1, 1) + RN = (1, 1) + \frac{5}{2}(1, -2) = (\frac{7}{2}, 4)$$

The circle is

$$(x - \frac{7}{2})^2 + (y + 4)^2 = \frac{125}{4}.$$
4. A baseball is thrown into the air from two meters above the origin with initial velocity 
\[40\hat{i} + 9\hat{k}\text{ (m/s)}.\] Assume the land is flat and \(\hat{k}\) is the vertical upward direction. The
baseball encounters three forces: gravity, air resistance, and Magnus force due to spin of
the ball in the air. For simplicity we assume these forces result in a constant
acceleration \(a = -10\hat{i} - \hat{j} - 10\hat{k}\text{ (m/s}^2\)). Where does the ball land and with what speed?

**Answer.**

\[
v(t) = v(0) + \int_0^t a(\tau)\,d\tau = (40 - 10t, -t, 9 - 10t).
\]

\[
r(t) = r(0) + \int_0^t v(\tau)\,d\tau = (40t - 5t^2, -t^2/2, 2 + 9t - 5t^2)
\]

It lands when \(z(t) = 2 + 9t - 5t^2 = 0\), i.e. \(t = 2\) or \(t = -1/5\). We take the positive value
\(t = 2\).

It lands at
\[
r(2) = (60, -2, 0)\text{ (m)},
\]
and, since \(v(2) = (20, -2, -11)\),

\[
|v(2)| = (20^2 + 2^2 + 11^2)^{1/2} = \sqrt{525} \approx 22.9\text{ (m/s)}
\]

**Remark 1.** The air resistance should be in the direction of \(-v(t)\). For baseball
dynamics see e.g. [http://spiff.rit.edu/richmond/baseball/traj/traj.html](http://spiff.rit.edu/richmond/baseball/traj/traj.html).

**Remark 2.** Many of you used the textbook formula \(a = -g\hat{k}\) which assumes gravity
only and no resistance nor Magnus force.

**Remark 3.** Although I did not deduct any point, velocity \(v\) is not speed \(|v|\).
5. Consider the vector field \( \mathbf{F} = (F_1, F_2) = (x + y) \mathbf{i} + (x - y) \mathbf{j} \).

(2 points) (a) Identify and sketch its nullclines, where \( F_1 = 0 \) (vertical vectors) or \( F_2 = 0 \) (horizontal vectors)

\[
\begin{align*}
x - y &= 0 \\
x + y &= 0
\end{align*}
\]

(6 points) (b) Sketch some of the vectors \( \mathbf{F} \) on its nullclines and the two axes.