Another application of the Divergence Theorem occurs in fluid flow. Let \( \mathbf{v}(x, y, z) \) be the velocity field of a fluid with constant density \( \rho \). Then \( \mathbf{F} = \rho \mathbf{v} \) is the rate of flow per unit area. If \( P_0(x_0, y_0, z_0) \) is a point in the fluid and \( B_a \) is a ball with center \( P_0 \) and very small radius \( a \), then \( \text{div} \ \mathbf{F}(P) \approx \text{div} \ \mathbf{F}(P_0) \) for all points in \( B_a \) since \( \text{div} \ \mathbf{F} \) is continuous. We approximate the flux over the boundary sphere \( S_a \) as follows:

\[
\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \text{div} \ \mathbf{F} \ dV \approx \iiint_{B_a} \text{div} \ \mathbf{F}(P_0) \ dV = \text{div} \ \mathbf{F}(P_0) V(B_a)
\]

This approximation becomes better as \( a \to 0 \) and suggests that

\[
\text{div} \ \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iiint_{S_a} \mathbf{F} \cdot d\mathbf{S}
\]

Equation 8 says that \( \text{div} \ \mathbf{F}(P_0) \) is the net rate of outward flux per unit volume at \( P_0 \). (This is the reason for the name divergence.) If \( \text{div} \ \mathbf{F}(P) > 0 \), the net flow is outward near \( P \) and \( P \) is called a source. If \( \text{div} \ \mathbf{F}(P) < 0 \), the net flow is inward near \( P \) and \( P \) is called a sink.

For the vector field in Figure 4, it appears that the vectors that end near \( P_1 \) are shorter than the vectors that start near \( P_1 \). Thus the net flow is outward near \( P_1 \), so \( \text{div} \ \mathbf{F}(P_1) > 0 \) and \( P_1 \) is a source. Near \( P_2 \), on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so \( \text{div} \ \mathbf{F}(P_2) < 0 \) and \( P_2 \) is a sink. We can use the formula for \( F \) to confirm this impression. Since \( \mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} \), we have \( \text{div} \ \mathbf{F} = 2x + 2y \), which is positive when \( y > -x \). So the points above the line \( y = -x \) are sources and those below are sinks.

16.9 Exercises

1-4 Verify that the Divergence Theorem is true for the vector field \( \mathbf{F} \) on the region \( E \).

1. \( \mathbf{F}(x, y, z) = 3x \mathbf{i} + xy \mathbf{j} + 2xz \mathbf{k} \),
   \( E \) is the cube bounded by the planes \( x = 0, x = 1, y = 0, y = 1, z = 0, \) and \( z = 1 \)

2. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k} \),
   \( E \) is the solid bounded by the paraboloid \( z = 4 - x^2 - y^2 \) and the \( xy \)-plane

3. \( \mathbf{F}(x, y, z) = (z, y, x) \),
   \( E \) is the solid ball \( x^2 + y^2 + z^2 = 16 \)

4. \( \mathbf{F}(x, y, z) = (x^2, y^2, z) \),
   \( E \) is the solid cylinder \( y^2 + z^2 \leq 9, 0 \leq x \leq 2 \)

5-15 Use the Divergence Theorem to calculate the surface integral \( \iint_{S} \mathbf{F} \cdot d\mathbf{S} \), that is, calculate the flux of \( \mathbf{F} \) across \( S \).

5. \( \mathbf{F}(x, y, z) = xz^2 \mathbf{i} + yz^2 \mathbf{j} - ye^z \mathbf{k} \),
   \( S \) is the surface of the box bounded by the coordinate planes and the planes \( x = 3, y = 2, \) and \( z = 1 \)

6. \( \mathbf{F}(x, y, z) = x^2yz \mathbf{i} + xy^2z \mathbf{j} + yz^2 \mathbf{k} \),
   \( S \) is the surface of the box enclosed by the planes \( x = 0, x = a, y = 0, y = b, z = 0, \) and \( z = c \), where \( a, b, \) and \( c \) are positive numbers

7. \( \mathbf{F}(x, y, z) = 3xy^2 \mathbf{i} + xe^y \mathbf{j} + x^3 \mathbf{k} \),
   \( S \) is the surface of the solid bounded by the cylinder \( y^2 + z^2 = 1 \) and the planes \( x = -1 \) and \( x = 2 \)

8. \( \mathbf{F}(x, y, z) = (x^3 + y^3) \mathbf{i} + (y^3 + z^3) \mathbf{j} + (z^3 + x^3) \mathbf{k} \),
   \( S \) is the sphere with center the origin and radius 2

9. \( \mathbf{F}(x, y, z) = x^2 \sin y \mathbf{i} + x \cos y \mathbf{j} - z \sin y \mathbf{k} \),
   \( S \) is the “fat sphere” \( x^2 + y^2 + z^2 = 8 \)

10. \( \mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + z \mathbf{k} \),
   \( S \) is the surface of the tetrahedron enclosed by the coordinate planes and the plane
   \[
   \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1
   \]
   where \( a, b, \) and \( c \) are positive numbers

11. \( \mathbf{F}(x, y, z) = (\cos z + xy^2) \mathbf{i} + xe^y \mathbf{j} + (\sin y + x^2z) \mathbf{k} \),
   \( S \) is the surface of the solid bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 4 \)

12. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} - x^2 y^2 \mathbf{j} + 4xy^2z \mathbf{k} \),
   \( S \) is the surface of the solid bounded by the cylinder \( x^2 + y^2 = 1 \) and the planes \( z = x + 2 \) and \( z = 0 \)

13. \( \mathbf{F} = |r| \mathbf{r} \), where \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \),
   \( S \) consists of the hemisphere \( z = \sqrt{1 - x^2 - y^2} \) and the disk \( x^2 + y^2 \leq 1 \) in the \( xy \)-plane

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CAS Computer algebra system required

1. Homework Hints available at stewartcalculus.com
14. \( \mathbf{F} = |r|^2 \mathbf{r} \), where \( r = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \),
   \( S \) is the sphere with radius \( R \) and center the origin.

15. \( \mathbf{F}(x, y, z) = e^z \tan z \mathbf{i} + y \sqrt{3} - x^2 \mathbf{j} + x \sin y \mathbf{k} \),
   \( S \) is the surface of the solid that lies above the \( xy \)-plane
   and below the surface \( z = 2 - x^2 - y^2 \), \(-1 \leq x \leq 1, -1 \leq y \leq 1 \).

16. Use a computer algebra system to plot the vector field
   \( \mathbf{F}(x, y, z) = \sin x \cos^3 y \mathbf{i} + \sin^3 y \cos z \mathbf{j} + \sin z \cos^2 x \mathbf{k} \)
   in the cube cut from the first octant by the planes \( x = \pi/2 \),
   \( y = \pi/2 \), and \( z = \pi/2 \). Then compute the flux across the
   surface of the cube.

17. Use the Divergence Theorem to evaluate \( \iint_S \mathbf{F} \cdot d\mathbf{S} \), where
   \( \mathbf{F}(x, y, z) = z^2 x \mathbf{i} + \left( \frac{1}{2} y^2 + \tan z \right) \mathbf{j} + (x^2 + y^2)^3 \mathbf{k} \)
   and \( S \) is the top half of the sphere \( x^2 + y^2 + z^2 = 1 \).
   \( \text{[Hint: Note that } S \text{ is not a closed surface. First compute}
   \text{integrals over } S_1 \text{ and } S_2, \text{ where } S_1 \text{ is the disk}
   \text{ } x^2 + y^2 < 1, \text{ oriented downward, and } S_2 = S \cup S_1.] \)

18. Let \( \mathbf{F}(x, y, z) = 2 \tan^{-1} (y^2) \mathbf{i} + z^2 \ln(x^2 + 1) \mathbf{j} + z \mathbf{k} \).
   Find the flux of \( \mathbf{F} \) across the part of the paraboloid
   \( x^2 + y^2 + z = 2 \) that lies above the plane \( z = 1 \) and is
   oriented upward.

19. A vector field \( \mathbf{F} \) is shown. Use the interpretation of divergence
derived in this section to determine whether \( \text{div } \mathbf{F} \)
   is positive or negative at \( P_1 \) and at \( P_2 \).

20. (a) Are the points \( P_1 \) and \( P_2 \) sources or sinks for the vector
    field \( \mathbf{F} \) shown in the figure? Give an explanation based
    solely on the picture.
    (b) Given that \( \mathbf{F}(x, y) = (x, y^2) \), use the definition of divergence
    to verify your answer to part (a).

21. \( \mathbf{F}(x, y) = (xy, x + y^2) \)
22. \( \mathbf{F}(x, y) = (x^2, y^3) \)

23. Verify that \( \text{div } \mathbf{E} = 0 \) for the electric field \( \mathbf{E}(x) = \frac{eQ}{|x|^3} \mathbf{x} \).

24. Use the Divergence Theorem to evaluate
   \( \iiint \int_S (2x + 2y + z^2) \,dS \)
   where \( S \) is the sphere \( x^2 + y^2 + z^2 = 1 \).

25–30 Prove each identity, assuming that \( S \) and \( E \) satisfy
   the conditions of the Divergence Theorem and the scalar functions
   and components of the vector fields have continuous second-
   order partial derivatives.

25. \( \iint_S \mathbf{a} \cdot \mathbf{n} \,dS = 0 \), where \( \mathbf{a} \) is a constant vector

26. \( \nabla \mathbf{E} = \frac{1}{r} \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \)

27. \( \iiint_S \mathbf{F} \cdot d\mathbf{S} = 0 \)
28. \( \iiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_E \nabla^2 \mathbf{f} \,dV \)

29. \( \iiint_S (f \nabla g) \cdot \mathbf{n} \,dS = \iiint_E (f \nabla^2 g + \nabla f \cdot \nabla g) \,dV \)
30. \( \iiint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \,dS = \iiint_E (f \nabla^2 g - g \nabla^2 f) \,dV \)

31. Suppose \( S \) and \( E \) satisfy the conditions of the Divergence
    Theorem and \( f \) is a scalar function with continuous partial
    derivatives. Prove that

    \[ \iint_S f \mathbf{n} \,dS = \iiint_E \nabla f \,dV \]

    These surface and triple integrals of vector functions are
    vectors defined by integrating each component function.
    \( \text{[Hint: Start by applying the Divergence Theorem to } \mathbf{F} = cf, \)
    where \( c \) is an arbitrary constant vector. \]

32. A solid occupies a region \( E \) with surface \( S \) and is immersed
    in a liquid with constant density \( \rho \). We set up a coordinate
    system so that the \( xy \)-plane coincides with the surface of
    the liquid, and positive values of \( z \) are measured downward
    into the liquid. Then the pressure at depth \( z \) is \( p = \rho g z \),
    where \( g \) is the acceleration due to gravity (see Section 8.3). The total
    buoyant force on the solid due to the pressure distribution is
given by the surface integral

    \[ \mathbf{F} = -\iint_S \rho \mathbf{n} \,dS \]

    where \( \mathbf{n} \) is the outer unit normal. Use the result of Exercise 31 to show
    that \( \mathbf{F} = -W \mathbf{k} \), where \( W \) is the weight of
    the liquid displaced by the solid. (Note that \( \mathbf{F} \) is directed
    upward because \( z \) is directed downward.) The result is
    Archimedes' Principle: The buoyant force on an object
    equals the weight of the displaced liquid.