1. The figure shows a curve \( C \) given by a vector function \( \mathbf{r}(t) \).
   (a) Draw the vectors \( \mathbf{r}(4.5) - \mathbf{r}(4) \) and \( \mathbf{r}(4.2) - \mathbf{r}(4) \).
   (b) Draw the vectors \( \mathbf{r}(4.5) - \mathbf{r}(4) \) and \( \mathbf{r}(4.2) - \mathbf{r}(4) \) given by
   \[
   \begin{align*}
   \frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}.
   \end{align*}
   \]
   (c) Write expressions for \( \mathbf{r}'(4) \) and the unit tangent vector \( \mathbf{T}(4) \).
   (d) Draw the vector \( \mathbf{T}(4) \).

2. (a) Make a large sketch of the curve described by the vector function \( \mathbf{r}(t) = (t^2, t^3) \), \( 0 \leq t \leq 2 \), and draw the vectors \( \mathbf{r}(1) \), \( \mathbf{r}(1.1) \), and \( \mathbf{r}(1.1) - \mathbf{r}(1) \).
   (b) Draw the vector \( \mathbf{r}'(1) \) starting at \( (1, 1) \), and compare it with the vector
   \[
   \frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}.
   \]
   Explain why these vectors are so close to each other in length and direction.

3. Sketch the plane curve with the given vector equation.
   (b) Find \( \mathbf{r}'(t) \).
   (c) Sketch the position vector \( \mathbf{r}(t) \) and the tangent vector \( \mathbf{r}'(t) \) for the given value of \( t \).

4. \( \mathbf{r}(t) = (t - 2, t^2 + 1), \quad t \in \mathbb{R} \)
5. \( \mathbf{r}(t) = (t^2, t^3), \quad t = 1 \)
6. \( \mathbf{r}(t) = \sin t \mathbf{i} + 2 \cos t \mathbf{j}, \quad t = \pi/2 \)
7. \( \mathbf{r}(t) = e^t \mathbf{i} + e^t \mathbf{j}, \quad t = 0 \)
8. \( \mathbf{r}(t) = (1 + \cos t) \mathbf{i} + (2 + \sin t) \mathbf{j}, \quad t = \pi/2 \)

9. Find the derivative of the vector function.
   (a) \( \mathbf{r}(t) = (t \sin t, t^2, t \cos 2t) \)
   (b) \( \mathbf{r}(t) = (\tan t, \sec t, 1/t^2) \)
   (c) \( \mathbf{r}(t) = t \mathbf{i} + t \mathbf{j} + 2 \sqrt{t} \mathbf{k} \)
   (d) \( \mathbf{r}(t) = \frac{1}{1 + t} \mathbf{i} + \frac{t}{1 + t} \mathbf{j} + \frac{t^2}{1 + t} \mathbf{k} \)

10. Find the unit tangent vector \( \mathbf{T}(t) \) at the point with the given value of the parameter \( t \).
   17. \( \mathbf{r}(t) = (te^t, 2 \arctan t, 2e^t), \quad t \in \mathbb{R} \)
   18. \( \mathbf{r}(t) = (t^2 + 3t, t^2 + 1, 3t + 4), \quad t = 1 \)
   19. \( \mathbf{r}(t) = \cos t \mathbf{i} + 3t \mathbf{j} + 2 \sin 2t \mathbf{k}, \quad t = 0 \)
   20. \( \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + \tan^2 t \mathbf{k}, \quad t = \pi/4 \)

21. If \( \mathbf{r}(t) = (t, t^2, t^3) \), find \( \mathbf{r}'(t), \mathbf{T}(1), \mathbf{r}''(t), \mathbf{r}'(t) \times \mathbf{r}''(t) \).
22. If \( \mathbf{r}(t) = (e^t, e^{-t}, te^t) \), find \( \mathbf{T}(0), \mathbf{r}'(0), \mathbf{r}'(t) \times \mathbf{r}'(t) \).

23. Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.
   23. \( x = 1 + 2t, \quad y = 3 - t, \quad z = 5 + t; \quad (3, 0, 2) \)
   24. \( x = e^{-t}, \quad y = te^{-t}, \quad z = te^{-t}; \quad (1, 0, 0) \)
   25. \( x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad z = e^{-t}; \quad (1, 0, 1) \)
   26. \( x = t^2 + 3, \quad y = \ln (t^2 + 3), \quad z = t; \quad (2, 4, 1) \)

27. Find a vector function for the tangent line to the curve of intersection of the cylinders \( x^2 + y^2 = 25 \) and \( x^2 + z^2 = 20 \) at the point \( (3, 4, 2) \).
28. Find the point on the curve \( \mathbf{r}(t) = (2 \cos t, 2 \sin t, e^t), \quad 0 \leq t \leq \pi \) where the tangent line is parallel to the plane \( 3x + y = 1 \).

29. Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.
   29. \( x = t, \quad y = e^{-t}, \quad z = 2t - t^3; \quad (0, 1, 0) \)
   30. \( x = 2 \cos t, \quad y = 2 \sin t, \quad z = 4 \cos 2t; \quad (\sqrt{3}, 1, 2) \)
   31. \( x = t \cos t, \quad y = t, \quad z = t \sin t; \quad (0, \pi, 0) \)

32. (a) Find the point of intersection of the tangent lines to the curve \( r(t) = (\sin t, 2 \sin t, \cos t) \) at the points where \( t = 0 \) and \( t = 0.5 \).
   (b) Illustrate by graphing the curve and both tangent lines.
33. The curves \( r_1(t) = (t, t^2, t^3) \) and \( r_2(t) = (\sin t, \sin 2t, t) \) intersect at the origin. Find their angle of intersection correct to the nearest degree.
34. At what point do the curves \( r(t) = (t, 1 - t, 3 + t^2) \) and \( r(s) = (3 - s, s - 2, s^2) \) intersect? Find their angle of intersection correct to the nearest degree.

35–40 Evaluate the integral.

35. \( \int_0^1 (t^2 - t^3) \mathbf{j} + 3t^2 \mathbf{k} \, dt \)

36. \( \int_0^1 \left( \frac{4}{1 + t^2} \mathbf{j} + \frac{2t}{1 + t^2} \mathbf{k} \right) \, dt \)

37. \( \int_0^{\pi/2} (3 \sin^2 t \cos t \mathbf{i} + 3 \sin t \cos^2 t \mathbf{j} + 2 \sin t \cos t \mathbf{k}) \, dt \)

38. \( \int_0^1 (t^2 \mathbf{i} + t \sqrt{3 - 1} \mathbf{j} + t \sin \pi t \mathbf{k}) \, dt \)

39. \( \int_0^1 (\sec^2 t \mathbf{i} + t(t^2 + 1) \mathbf{j} + t \ln t \mathbf{k}) \, dt \)

40. \( \int_0^1 \left( e^{2t} \mathbf{i} + \frac{t}{1 - t} \mathbf{j} + \frac{1}{\sqrt{1 - t^2}} \mathbf{k} \right) \, dt \)

41. Find \( r(t) \) if \( r'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j} + \sqrt{t} \mathbf{k} \) and \( r(1) = \mathbf{i} + \mathbf{j} \).

42. Find \( r(t) \) if \( r'(t) = t \mathbf{i} + e^t \mathbf{j} + t e^t \mathbf{k} \) and \( r(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \).

43. Prove Formula 1 of Theorem 3.

44. Prove Formula 3 of Theorem 3.

45. Prove Formula 5 of Theorem 3.

46. Prove Formula 6 of Theorem 3.

47. If \( u(t) = (t, \cos t, \sin t) \) and \( v(t) = (t, \cos t, \sin t) \), use Formula 4 of Theorem 3 to find

\[ \frac{d}{dt} [u(t) \cdot v(t)] \]

48. If \( u \) and \( v \) are the vector functions in Exercise 47, use Formula 5 of Theorem 3 to find

\[ \frac{d}{dt} [u(t) \times v(t)] \]

49. Find \( f''(2) \), where \( f(t) = u(t) \cdot v(t) \), \( u(2) = (1, 2, -1) \), \( u'(2) = (3, 0, 4) \), and \( v(t) = (t, t', t'^2) \).

50. If \( r(t) = u(t) \times v(t) \), where \( u \) and \( v \) are the vector functions in Exercise 49, find \( r''(2) \).

51. Show that if \( r \) is a vector function such that \( r'' \) exists, there is

\[ \frac{d}{dt} [r(t) \times r'(t)] = r(t) \times r''(t) \]

52. Find an expression for \( \frac{d}{dt} [u(t) \cdot (v(t) \times w(t))] \).

53. If \( r(t) \neq 0 \), show that

\[ \frac{d}{dt} |r(t)| = \frac{1}{|r(t)|} \cdot r(t) \cdot r'(t). \]

[Hint: \( |r(t)|^2 = r(t) \cdot r(t)\)]

54. If a curve has the property that the position vector \( r(t) \) is always perpendicular to the tangent vector \( r'(t) \), show that the curve lies on a sphere with center the origin.

55. If \( u(t) = r(t) \cdot [r'(t) \times r''(t)] \), show that

\[ u'(t) = r(t) \cdot [r'(t) \times r'''(t)] \]

56. Show that the tangent vector to a curve defined by a vector function \( r(t) \) points in the direction of increasing \( t \). [Hint: Refer to Figure 1 and consider the cases \( h > 0 \) and \( h < 0 \) separately.]

13.3 Arc Length and Curvature

In Section 10.2 we defined the length of a plane curve with parametric equations \( x = f(t) \), \( y = g(t) \), \( a \leq t \leq b \), as the limit of lengths of inscribed polygons and, for the case where \( f' \) and \( g' \) are continuous, we arrived at the formula

\[ L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt = \int_a^b \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

The length of a space curve is defined in exactly the same way (see Figure 1). Suppose that the curve has the vector equation \( r(t) = (f(t), g(t), h(t)) \), \( a \leq t \leq b \), or, equivalently, the parametric equations \( x = f(t), y = g(t), z = h(t) \), where \( f', g', \) and \( h' \) are continuous. If the curve is traversed exactly once as \( t \) increases from \( a \) to \( b \), then it can be shown that its length is

\[ L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt \]

\[ = \int_a^b \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt \]