Family Name ___________________________  Given Name ___________________________

Student Number ______________________  Signature ____________________________

No calculators, books, notebooks or any other written materials are allowed

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Rules for the exam:

1. Bring a photo ID for the inspection of the invigilator.

2. During the exam, people may be relocated, for many possible reasons.

3. From five minutes before the end of the exam, you cannot hand in your exam any more and should wait in your seat until the end of the exam.

4. When the invigilator says that the exam is over, you should stop writing and remain seated. Please pass your exam to the nearest aisle.

5. Do not discuss before you leave the room, since your neighbor may change her/his solutions after hearing your conversation.

6. You are not allowed to leave until the invigilator has collected all exams and says that you can leave.
1. The vector field \( \mathbf{F}(x, y, z) = (Ax^3 y + 2xz^3)\mathbf{i} + (x^4 - 3y^2z^2)\mathbf{j} + (-2y^3z + Bx^2z^2)\mathbf{k} \) is conservative on \( \mathbb{R}^3 \).

(a) (4 pt) Find the values of the constants \( A \) and \( B \).

\[ \text{Solution.} \quad \text{We have} \]
\[ 0 = \text{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (Ax^3y + 2xz^3) & (x^4 - 3y^2z^2) & (-2y^3z + Bx^2z^2) \end{vmatrix} = (-6y^2z - (-6y^2z), 6xz^2 - 2Bxz^2, 4x^3 - Ax^3) \] (2pt)

Thus
\[ A = 4, \quad B = 3. \quad (2pt) \]

(b) (4 pt) Find a potential function \( \phi \) such that \( \mathbf{F} = \nabla \phi \) on \( \mathbb{R}^3 \) and \( \phi(0, 0, 0) = 3 \).

\[ \text{Solution.} \quad \text{Denote} \quad \mathbf{F} = (P, Q, R). \quad \text{Since} \quad \phi_x = P, \quad \text{we get} \]
\[ \phi = \int P(x, y, z)dx = \int (4x^3y + 2xz^3)dx = x^4y + x^2z^3 + f(y, z). \quad (1pt) \]

We have \( \phi_y = x^4 + f_y(y, z) = Q = x^4 - 3y^2z^2 \), thus
\[ f_y(y, z) = -3y^2z^2, \quad f(y) = \int (-3y^2z^2)dy = -y^3z^2 + g(z). \quad (1pt) \]

We have \( \phi_z = 3x^2z^2 - 2y^3z + g'(z) = R = -2y^3z + 3x^2z^2 \), thus
\[ g'(z) = 0, \quad g(z) = C \quad (1pt) \]

for some constant \( C \) and \( \phi(x, y) = x^4y + x^2z^3 - y^3z^2 + C \). Since \( \phi(0, 0, 0) = 3 \), we have \( C = 3 \) and
\[ \phi(x, y) = x^4y + x^2z^3 - y^3z^2 + 3. \quad (1pt) \]

(c) (2 pt) If \( C \) is the curve \( \mathbf{F}(t) = (t^{4567}, t^{5678}, t^{6789}) \) from \((-1, 1, -1)\) to \((1, 1, 1)\), evaluate
\[ I = \int_C \mathbf{F} \cdot d\mathbf{r}. \]

\[ \text{Solution.} \quad \text{Since} \quad \mathbf{F} \quad \text{is conservative,} \]
\[ I = \phi(1, 1, 1) - \phi(-1, 1, -1) = (1 + 1 - 1 + 3) - (1 - 1 - 1 + 3) = 2. \quad (1pt) \]

\[ \text{Note:} \quad \text{The points correspond to} \quad t = -1 \quad \text{and} \quad t = 1. \quad \text{You can also compute it by} \]
\[ I = \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt. \]

\[ \text{Instructor note.} \quad \text{If they tried to compute} \quad \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt \quad \text{instead, I did not give any points if they made an error setting up the integral.} \]
2. Let \( \vec{F}(x, y) = \frac{y^3}{(x^2 + y^2)^2} \hat{i} - \frac{xy^2}{(x^2 + y^2)^2} \hat{j} \) for \( (x, y) \neq (0, 0) \).

(a) Compute \( I_1 = \int_{C_1} \vec{F} \cdot d\vec{r} \) where \( C_1 \) is the unit circle in the \( xy \)-plane, positively oriented.

**Solution.** Parametrize \( C_1 \) by \( \vec{r}(t) = (\cos t, \sin t) \) with \( 0 \leq t \leq 2\pi \). (1pt) We have

\[
\vec{F}(\vec{r}(t)) = (\sin^3 t, -\cos t \sin^2 t),
\]
\[
\vec{r}'(t) = (-\sin t, \cos t). \] (1pt)

Thus

\[
\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = -\sin^4 t - \cos^2 t \sin^2 t = -\sin^2 t \] (2pt)

and

\[
I_1 = \int_{0}^{2\pi} (-\sin^2 t) dt = -\pi. \] (1pt)

**Instructor note.** Many students tried to compute the line integral from Green’s theorem: \( \int_{C_1} \vec{F} \cdot d\vec{r} = \iint_{\text{disk}} (Q_x - P_y) dA \) (forgetting that this is invalid because \( F \) is singular at the origin). If this was the case, I awarded 0, 1, or 2 pts, depending on if they attempted the line integral or not.

Since the curve \( C_1 \) is a nice circle, not a weird curve like an ellipse in part (b), one should try calculating the line integral directly before trying Green’s theorem.

(b) Assume you know the value of \( I_1 \). Use Green’s theorem to find \( I_2 = \int_{C_2} \vec{F} \cdot d\vec{r} \) in terms of \( I_1 \), where \( C_2 \) is the ellipse \( \frac{x^2}{16} + \frac{y^2}{25} = 1 \), positively oriented.

**Solution.** Denote \( \vec{F} = (P, Q) \) and \( A = x^2 + y^2 \). We have

\[
Q_x - P_y = (-\frac{xy^2}{A^2})_x - (\frac{y^3}{A^2})_y
= -\frac{y^2}{A^2} + \frac{2xy^2}{A^3} \cdot 2x - \frac{3y^2}{A^2} + \frac{2y^3}{A^3} \cdot 2y = 0. \] (2pt)

Let \( E \) denote the region inside \( C_2 \) and outside \( C_1 \). Note that \( C_1 \) is negatively oriented with respect to \( E \). By Green’s theorem,

\[
-I_1 + I_2 = \int_{E} (Q_x - P_y) dA = 0. \] (2pt)

Thus

\[
I_2 = I_1 \] (1pt)

**Remark.** \( \vec{F} = \nabla \phi \) with \( \phi = -\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \). Note that \( \phi \) and \( \theta \) are multi-valued in \( D = \mathbb{R}^2 \setminus \{0\} \) but \( \sin 2\theta = 2 \cos \theta \sin \theta = \frac{2xy}{x^2 + y^2} \) is single-valued.

**Instructor note.** Many students made mistakes computing \( Q_x - P_y \). Again (like in part (a)), many students made the error writing \( I_2 = \int_{\text{ellipse}} (Q_x - P_y) dA \).
3. (a) (5 pt) A surface $S$ has the parametric equation $\vec{r} = (u^2 + 1, v^3 + 1, u + v)$. Find its tangent plane at the point $P(5, 2, 3)$.

**Solution.** This is §16.6 #34. At $P(5, 2, 3)$,

$$u^2 + 1 = 5, \quad v^3 + 1 = 2, \quad u + v = 3.$$ 

Thus $v = 1$ since $v^3 + 1 = 2$, and $u = 2$ since $u + v = 3$. Hence $P = \vec{r}(2, 1)$. (2pt)

We have

$$\vec{r}_u(P) = (2u, 0, 1)|_P = (4, 0, 1), \quad \vec{r}_v(P) = (0, 3v^2, 1)|_P = (0, 3, 1).$$

Hence

$$\vec{r}_u \times \vec{r}_v(P) = (-3, -4, 12).$$

The tangent plane at $P$ has equation

$$-3(x - 5) - 4(y - 2) + 12(z - 3) = 0,$$

or $-3x - 4y + 12z = 13$. (1pt)

(b) (5 pt) Find the area of the cone $x = \sqrt{y^2 + z^2}$ that lies between the plane $y = z$ and the cylinder $y = z^2$.

**Solution.** This is §16.6 #42 with $x$ and $z$ switched.

The plane and the cylinder intersect at the lines $y = z = 0$ and $y = z = 1$. The surface is a graph $x = f(y, z) = \sqrt{y^2 + z^2}$ with domain

$$D : 0 \leq z \leq 1, \quad z^2 \leq y \leq z.$$ 

(1pt)

We have

$$f_y = \frac{y}{f}, \quad f_z = \frac{z}{f},$$

$$\sqrt{1 + f_y^2 + f_z^2} = \sqrt{1 + y^2/f^2 + z^2/f^2} = \sqrt{2}.$$ 

Thus the area is

$$A(S) = \int \int_D \sqrt{1 + f_y^2 + f_z^2} \, dy \, dz = \int \int_D \sqrt{2} \, dy \, dz$$ 

(1pt)

$$= \int_0^1 \int_{z^2}^z \sqrt{2} \, dy \, dz$$ 

(1pt)

$$= \sqrt{2} \int_0^1 \left( z - z^2 \right) \, dz = \sqrt{2} \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\sqrt{2}}{6}.$$ 

(1pt)

**Solution #2.** $A(S) = \int \int_D \sqrt{1 + f_y^2 + f_z^2} \, dy \, dz = \int_0^1 \left( f_y \sqrt{2} \, dz \right) \, dy = \frac{\sqrt{2}}{6}$.
4. Circle the correct answers. No justifications necessary.

(a) Let the vector fields $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ be as shown in the graph. Let $C$ denote the circle centered at $(2, 2)$ with radius 1, positively oriented. Find the signs of $I_j = \int_C \vec{F}_j \cdot d\vec{r}$, $j = 1, 2, 3$.

$I_1 < 0, \quad I_1 = 0, \quad I_1 > 0; \quad I_2 < 0, \quad I_2 = 0, \quad I_2 > 0; \quad I_3 < 0, \quad I_3 = 0, \quad I_3 > 0.

Solution. $I_1 = 0, \quad I_2 = 0, \quad I_3 < 0$ (Green Theorem and $Q_x - P_y = 0, = 0, < 0$.)

(b) True or false? $\nabla \times (\nabla f) = 0$ for all $C^2$ scalar function $f$ in $\mathbb{R}^3$.

Solution. True

(c) True or false? $\nabla \times (f \nabla f) = 0$ for all $C^2$ scalar function $f$ in $\mathbb{R}^3$.

Solution. True (note $f \nabla f = \nabla (f^2/2)$ and use (b), you can also use $\nabla \times (f \vec{F}) = \nabla f \times \vec{F} + f \nabla \times \vec{F}$)

(d) True or false? $\text{div}(f \vec{F}) = \nabla f \cdot \vec{F} + f \text{div} \vec{F}$ for all $C^1$ scalar function $f$ and $C^1$ vector field $\vec{F}$ in $\mathbb{R}^3$.

Solution. True

(e) True or false? If a vector field $\vec{F} = (P, Q)$ in $\mathbb{R}^2$ has $Q = 0$, everywhere in $\mathbb{R}^2$, then the line integral $\int_C \vec{F} \cdot d\vec{r}$ is zero, for every simple closed curve $C$ in $\mathbb{R}^2$.

Solution. False (let $\vec{F} = (y, 0)$ and $C$ the square from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$.)

(f) True or false? The vector field $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is conservative in its domain $D$, which is $\mathbb{R}^2$ without the origin.

Solution. False (there is no potential in $D$)

(g) True or false? The vector field $\vec{F} = \left(-\frac{y}{x^2+y^2}\right)$ is conservative in the disk $(x + 2)^2 + y^2 < 1$.

Solution. True ($\vec{F} = \nabla \theta$ where $\theta$ is continuously defined on left half plane)
(h) True or false? The line integral \( \int_C \vec{F} \cdot d\vec{r} \) for the vector field \( \vec{F} = \left( \frac{-y}{x^2+y^2}, x \right) \) along the positively oriented circle \( C : (x+2)^2 + y^2 = 9 \) is 0.

**Solution.** False (It is \( 2\pi \) since \( C \) encloses the origin.)

**Instructor note.** True or False questions are good since they can test the student understanding of concepts, and they are not straightforward computation which is only part of the learning goal. For a midterm exam, 6-8 questions may be enough?

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**Formulas for MT2**

1. For a curve \( \vec{r}(t) \), \( s = \int_0^t |\vec{r}'(\tau)|d\tau, \quad \frac{ds}{dt} = |\vec{r}'|, \quad ds = |\vec{r}'(t)|dt \)

2. \( \mathbf{T} = \frac{\vec{r}'}{|\vec{r}'|}, \quad \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N} \)

3. \( \kappa = \frac{|d\mathbf{T}|}{ds} = \frac{|\mathbf{T}'|}{|\mathbf{r}'|^3}, \quad \kappa \mathbf{N} = \frac{d\mathbf{T}}{ds} \)

4. For \( y = f(x) \), \( \kappa(x) = \frac{|f''(x)|}{\sqrt{1+(f'(x))^2}^{3/2}} \)

5. Green’s theorem: \( \int_C Pdx + Qdy = \iint_D (Q_x - P_y) dA \)

6. For a surface \( S \) given by \( \vec{r}(u,v) : D \to \mathbb{R}^3 \), the surface area is \( \iint_D |\vec{r}_u \times \vec{r}_v| \, dudv \)

7. For a graph \( S \) given by \( z = f(x,y) \), \( (x,y) \in D \), the surface area is \( \iint_D \sqrt{1+f_x^2+f_y^2} \, dxdy \)

8. For a surface of revolution \( S : r = f(z), a \leq z \leq b \), the surface area is \( \iint_S \, dS = \int_a^b 2\pi f(z) \sqrt{1+[f'(z)]^2} \, dz \)