1. (§2.1#9) Let \( A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix} \). What value(s) of \( k \), if any, will make \( AB = BA \)?

**Solution.**

\[
AB = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix} = \begin{bmatrix} -7 & 18 + 3k \\ -4 & -9 + k \end{bmatrix}, \quad \text{while} \quad BA = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 12 \\ -6 - k & -9 + k \end{bmatrix}.
\]

Then \( AB = BA \) if and only if \( 18 + 3k = 12 \) and \( -4 = -6 - k \), which happens if and only if \( k = -2 \).

2. (§2.1#10) Let \( A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \), \( B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \), and \( C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} \). Verify that \( AB = AC \) and yet \( B \neq C \).

**Solution.**

\[
AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}, \quad AC = \begin{bmatrix} 3 & -6 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}
\]

3. (§2.1#15) The following questions concern arbitrary matrices \( A, B, \) and \( C \) for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.

(a) If \( A \) and \( B \) are 2 \( \times \) 2 matrices with columns \( \vec{a}_1, \vec{a}_2 \) and \( \vec{b}_1, \vec{b}_2 \), respectively, then \( AB = [\vec{a}_1 \vec{b}_1, \vec{a}_2 \vec{b}_2] \).

(b) Each column of \( AB \) is a linear combination of the columns of \( B \) using weights from the corresponding column of \( A \).

(c) \( AB + AC = A(B + C) \)

(d) \( A^T + B^T = (A + B)^T \)

(e) The transpose of a product of matrices equals the product of their transposes in the same order.

**Solution.**

a. False. See the definition of \( AB \).

b. False. The roles of \( A \) and \( B \) should be reversed in the second half of the statement. See the box after Example 3.

c. True. See Theorem 2(b), read right to left.

d. True. See Theorem 3(b), read right to left.

e. False. The phrase “in the same order” should be “in the reverse order.” See the box after Theorem 3.
4. (§2.1#21) Suppose the last column of $AB$ is entirely zeros but $B$ itself has no column of zeros. What can be said about the columns of $A$?

**Solution.** Let $\vec{b}_p$ be the last column of $B$. By hypothesis, the last column of $AB$ is zero. Thus $A\vec{b}_p = \vec{0}$. However, $\vec{b}_p \neq \vec{0}$ because $B$ has no columns of zeros. Thus, the equation $A\vec{b}_p = \vec{0}$ is a linear dependence relation among the columns of $A$, and so the columns of $A$ are linearly dependent.

5. (§2.1#22) Show that if the columns of $B$ are linearly dependent, then so are the columns of $AB$.

**Solution.** If the columns of $B$ are linearly dependent, then there exists a nonzero vector $\vec{x}$ such that $B\vec{x} = \vec{0}$. From this, $A(B\vec{x}) = (AB)\vec{x} = \vec{0}$ (by associativity). Since $\vec{x}$ is nonzero, the columns of $AB$ must be linearly dependent.

6. (§2.2#2) Find the inverse of the matrix $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$.

**Solution.**

$$
\begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}^{-1} = \frac{1}{15-16} \begin{bmatrix} 5 & -2 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}
$$

7. (§2.2#9) Mark each statement True of False. Justify each answer.

(a) In order for a matrix $B$ to be the inverse of $A$, the equations $AB = I$ and $BA = I$ must both be true.
(b) If $A$ and $B$ are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of $AB$.
(c) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then $A$ is invertible.
(d) If $A$ is an invertible $n \times n$ matrix, then the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^n$.
(e) (skip)

**Solution.**

a. True, by definition of invertible.

b. False. See Theorem 6(b).

c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.

d. True. This follows from Theorem 5, which also says that the solution of $Ax = b$ is unique, for each $b$.

e. True, by the box just before Example 6.

Part (e) is not covered and hence it is skipped.
8. (§2.2#31) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$, if it exists.

Solution.

$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

9. (§2.2#37) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$. Construct a $2 \times 3$ matrix $C$ (by trial and error) using only 1, $-1$, and 0 as entries, such that $CA = I_2$. Compute $AC$ and note that $AC \neq I_3$.

Solution.

There are many possibilities for $C$, but $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ is the only one whose entries are 1, $-1$, and 0. With only three possibilities for each entry, the construction of $C$ can be done by trial and error. This is probably faster than setting up a system of 4 equations in 6 unknowns. The fact that $A$ cannot be invertible follows from Exercise 25 in Section 2.1, because $A$ is not square.

10. (§2.3#11) In the following questions, all matrices are $n \times n$. Mark each statement True or False. Justify each answer.

(a) If the equation $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution, then $A$ is row equivalent to the $n \times n$ identity matrix.

(b) If the columns of $A$ span $\mathbb{R}^n$, then the columns are linearly independent.

(c) If $A$ is an $n \times n$ matrix, then the equation $A\overrightarrow{x} = \overrightarrow{b}$ has at least one solution for each $\overrightarrow{b}$ in $\mathbb{R}^n$.

(d) If the equation $A\overrightarrow{x} = \overrightarrow{0}$ has a nontrivial solution, then $A$ has fewer than $n$ pivot positions.

(e) If $A^T$ is not invertible, then $A$ is not invertible.

Solution.
a. True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).

b. True. If statement (h) of the IMT is true, then so is statement (e).

c. False. Statement (g) of the IMT is true only for invertible matrices.

d. True, by the IMT. If the equation \( Ax = 0 \) has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that \( A \) must have fewer than \( n \) pivot positions.

e. True, by the IMT. If \( A^T \) is not invertible, then statement (1) of the IMT is false, and hence statement (a) must also be false.

11. (§2.3#17) Can a square matrix with two identical columns be invertible? Why or why not?

**Solution.** If a square matrix has two identical columns then its columns are linearly dependent. By the Invertible Matrix Theorem, it cannot be invertible.

12. (§2.3#26) Explain why the columns of \( A^2 \) span \( \mathbb{R}^n \) whenever the columns of an \( n \times n \) matrix \( A \) are linearly independent.

**Solution.** If the columns of \( A \) are linearly independent, then since \( A \) is square, \( A \) is invertible by the Invertible Matrix Theorem. So \( A^2 \), which is the product of invertible matrices, is also invertible. By the Invertible Matrix Theorem, the columns of \( A \) span \( \mathbb{R}^n \).

13. (§2.5#7,9) Let \( A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \) and

\[
\vec{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}.
\]

(a) How many vectors are in \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \)?
(b) How many vectors are in \( \text{Col} \ A \)?
(c) Is \( \vec{p} \) in \( \text{Col} \ A \)? Why or why not?
(d) Is \( \vec{p} \) in \( \text{Nul} \ A \)? Why or why not?

**Solution.**

a. There are three vectors: \( v_1, v_2, \) and \( v_3 \) in the set \( \{v_1, v_2, v_3\} \).

b. There are infinitely many vectors in \( \text{Span}\{v_1, v_2, v_3\} = \text{Col} \ A \).

c. Deciding whether \( p \) is in \( \text{Col} \ A \) requires calculation:

\[
[A \ \ p] \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{2} & 2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & -4 & -10 & 14 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\]

The equation \( Ax = p \) has a solution, so \( p \) is in \( \text{Col} \ A \).
d.

To determine whether \( \mathbf{p} \) is in \( \text{Nul} \ A \), simply compute \( A\mathbf{p} \). Using \( A \) and \( \mathbf{p} \) as in Exercise 7,

\[
A\mathbf{p} = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix} \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}.
\]

Since \( A\mathbf{p} \neq \mathbf{0} \), \( \mathbf{p} \) is not in \( \text{Nul} \ A \).

Note. Geometrically, the matrix \( A \) gives a linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) that maps \( \vec{e}_1 \) to \( \vec{p}_1 \), \( \vec{e}_2 \) to \( \vec{p}_2 \), and \( \vec{e}_3 \) to \( \vec{p}_3 \). \( \text{Col} \ A = T(\mathbb{R}^3) \) is the range of \( T \). For our problem, \( \text{Col} \ A \) is a plane passing the origin, and Part (c) shows that \( \vec{p} \) is on the plane. On the other hand, \( \text{Nul} \ A \) is the set of all \( \vec{x} \) that is collapsed to the origin under \( T \). It is a line for our problem. Part (d) shows that \( \vec{p} \) is not on the line.

14. (§2.5#18) Determine if the set of vectors

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix},
\]

is a basis for \( \mathbb{R}^3 \). Justify the answer.

**Solution.**

No. Place the three vectors into a 3x3 matrix \( A \) and determine whether \( A \) is invertible:

\[
A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & -1 & 1 \\ -3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -4 & -4 \\ 0 & 11 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix}
\]

The matrix \( A \) has two pivots, so \( A \) is not invertible by the IMT and its columns do not form a basis for \( \mathbb{R}^3 \) (as pointed out in Example 5).

15. (§2.5#24) The matrix \( A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \end{bmatrix} \) has an echelon form \( \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \end{bmatrix} \). Find a basis for \( \text{Col} \ A \) and a basis for \( \text{Nul} \ A \).

**Solution.**
16. (§2.5#36) What can be said about Nul C when C is a 6 × 4 matrix with linearly independent columns?

**Solution.** If the columns of C are linearly independent, then the equation \( C\vec{x} = \vec{0} \) has only the trivial (zero) solution. That is, Nul \( C = \{ \vec{0} \} \).