1. (§1.1#12) Solve the system

\[
\begin{align*}
    x_1 - 5x_2 + 4x_3 &= -3 \\
    2x_1 - 7x_2 + 3x_3 &= -2 \\
    -2x_1 + x_2 + 7x_3 &= -1
\end{align*}
\]

**Solution.**

Replace R2 by R2 + (-2)R1 and replace R3 by R3 + (2)R1. Finally, replace R3 by R3 + (3)R2.

\[
\begin{bmatrix}
    1 & -5 & 4 & -3 \\
    2 & -7 & 3 & -2 \\
    -2 & 1 & 7 & -1
\end{bmatrix} \sim
\begin{bmatrix}
    1 & -5 & 4 & -3 \\
    0 & 3 & 4 & -4 \\
    0 & 3 & 5 & 4
\end{bmatrix} \sim
\begin{bmatrix}
    1 & -5 & 4 & -3 \\
    0 & 3 & -5 & 4 \\
    0 & 0 & 0 & 5
\end{bmatrix}
\]

The system is inconsistent, because the last row would require that 0 = 5 if there were a solution. The solution set is empty.

2. (§1.1#14) Solve the system

\[
\begin{align*}
    2x_1 &- 6x_3 = -8 \\
    x_2 + 2x_3 &= 3 \\
    3x_1 + 6x_2 - 2x_3 &= -4
\end{align*}
\]

**Solution.**

\[
\begin{array}{cccc}
2 & 0 & -6 & -8 \\
0 & 1 & 2 & 3 \\
3 & 6 & -2 & -4
\end{array} \sim
\begin{array}{cccc}
1 & 0 & -3 & -4 \\
0 & 1 & 2 & 3 \\
3 & 6 & -2 & -4
\end{array} \sim
\begin{array}{cccc}
1 & 0 & -3 & -4 \\
0 & 1 & 2 & 3 \\
0 & 0 & -5 & -10
\end{array} \sim
\begin{array}{cccc}
1 & 0 & -3 & -4 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2
\end{array}
\]

The solution is (2, -1, 2).

3. (§1.1#16) Determine if the system is consistent. Do not completely solve the system.

\[
\begin{align*}
    2x_1 &- 4x_4 = -10 \\
    3x_2 + 3x_3 &= 0 \\
    x_3 + 4x_4 &= -1 \\
    -3x_1 + 2x_2 + 3x_3 + x_4 &= 5
\end{align*}
\]
Solution.
First replace $R_4$ by $R_4 + (3/2)R_1$ and replace $R_4$ by $R_4 + (-2/3)R_2$. (One could also scale $R_1$ and $R_2$ before adding to $R_4$, but the arithmetic is rather easy keeping $R_1$ and $R_2$ unchanged.) Finally, replace $R_4$ by $R_4 + (-1)R_3$.

\[
\begin{bmatrix}
2 & 0 & 0 & -4 & -10 \\
0 & 3 & 3 & 0 & 0 \\
0 & 0 & 1 & 4 & -1 \\
-3 & 2 & 3 & 1 & 5
\end{bmatrix}
\sim
\begin{bmatrix}
2 & 0 & 0 & -4 & -10 \\
0 & 3 & 3 & 0 & 0 \\
0 & 0 & 1 & 4 & -1 \\
0 & 2 & 3 & -5 & -10
\end{bmatrix}
\sim
\begin{bmatrix}
2 & 0 & 0 & -4 & -10 \\
0 & 3 & 3 & 0 & 0 \\
0 & 0 & 1 & 4 & -1 \\
0 & 0 & 1 & -5 & -10
\end{bmatrix}
\sim
\begin{bmatrix}
2 & 0 & 0 & -4 & -10 \\
0 & 3 & 3 & 0 & 0 \\
0 & 0 & 1 & 4 & -1 \\
0 & 0 & 0 & -9 & -9
\end{bmatrix}
\]

The system is now in triangular form and has a solution. In fact, using the argument from Example 2, one can see that the solution is unique.

4. (§1.1#20) Determine the value(s) of $h$ such that the matrix is the augmented matrix of a consistent linear system.

\[
\begin{bmatrix}
1 & h & -5 \\
2 & -8 & 6
\end{bmatrix}
\]

Solution.
Write $c$ for $-8 - 2h$. If $c = 0$, that is, if $h = -4$, then the system has no solution, because 0 cannot equal 16. Otherwise, when $h \neq -4$, the system has a solution.

5. (§1.2#4) Row reduce the matrix to reduced echelon form. Circle the pivot positions in the final matrix, and list the pivot columns.

\[
\begin{bmatrix}
1 & 2 & 4 & 5 \\
2 & 4 & 5 & 4 \\
4 & 5 & 4 & 2
\end{bmatrix}
\]

Remark. Ignore “pivot positions in the original matrix”, which does not make sense.

Solution.

\[
\begin{bmatrix}
1 & 2 & 4 & 5 \\
2 & 4 & 5 & 4 \\
4 & 5 & 4 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 4 & 5 \\
0 & 0 & -3 & -6 \\
0 & 0 & -12 & -18
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 4 & 5 \\
0 & 0 & -3 & -6 \\
0 & 0 & -3 & -6
\end{bmatrix}
\]

Pivot cols 1, 2, and 3

Question. Does it make sense to consider the pivot positions of

\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\]?

6. (§1.2#8) Find the general solutions of the system whose augmented matrix is

\[
\begin{bmatrix}
1 & -3 & 0 & -5 \\
-3 & 7 & 0 & 9
\end{bmatrix}
\]
Solution.

\[
\begin{bmatrix}
1 & -3 & 0 & -5 \\
-3 & 7 & 0 & 9 \\
6 & -4 & 8 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & -3 & 0 & -5 \\
0 & -2 & 0 & -6 \\
0 & 1 & 0 & 3
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{1} & 0 & 0 & 4 \\
\frac{1}{0} & 0 & 0 & 3 \\
\frac{1}{0} & 0 & 0 & 3
\end{bmatrix}
\]

Corresponding system of equations:
\[x_1 = 4\]
\[x_2 = 3\]
\[x_3 \text{ is free}\]

The basic variables (corresponding to the pivot positions) are \(x_1\) and \(x_2\). The remaining variable \(x_3\) is free. Solve for the basic variables in terms of the free variable. In this particular problem, the basic variables do not depend on the value of the free variable.

General solution:
\[
\begin{align*}
\begin{cases}
x_1 &= 4 \\
x_2 &= 3 \\
x_3 &= \text{free}
\end{cases}
\end{align*}
\]

7. (§1.2#11) Find the general solutions of the system whose augmented matrix is

\[
\begin{bmatrix}
3 & -2 & 4 & 0 \\
9 & -6 & 12 & 0 \\
6 & -4 & 8 & 0
\end{bmatrix}
\]

Solution.

\[
\begin{bmatrix}
3 & -2 & 4 & 0 \\
9 & -6 & 12 & 0 \\
6 & -4 & 8 & 0
\end{bmatrix}
= \begin{bmatrix}
3 & -2 & 4 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{1} & -2/3 & 4/3 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Corresponding system:
\[
\begin{align*}
\begin{cases}
x_1 - \frac{2}{3}x_2 + \frac{4}{3}x_3 &= 0 \\
0 &= 0 \\
0 &= 0
\end{cases}
\end{align*}
\]

Basic variable: \(x_1\); free variables \(x_2, x_3\). General solution:
\[
\begin{align*}
x_1 &= \frac{2}{3}x_2 - \frac{4}{3}x_3 \\
x_2 &= \text{free} \\
x_3 &= \text{free}
\end{align*}
\]

8. (§1.2#24) Suppose a system of linear equations has a \(3 \times 5\) augmented matrix whose fifth column is not a pivot column. Is the system consistent? Why (or why not)?

Solution.

The system is consistent because there is not a pivot in column 5, which means that there is not a row of the form \([0 \ 0 \ 0 \ 0 \ 1]\). Since the matrix is the augmented matrix for a system, Theorem 2 shows that the system has a solution.

9. (§1.2#26) Suppose a \(3 \times 5\) coefficient matrix for a linear system has three pivot columns. Is the system consistent? Why or why not?

Solution. Since the coefficient matrix has three pivot columns, there is a pivot in each row of the coefficient matrix. Thus the augmented matrix will not have a row of the form \([0 \ 0 \ 0 \ 0 \ 1]\), and the system is consistent.
10. (§1.3#11) Determine if \( \mathbf{b} \) is a linear combination of \( \mathbf{a}_1 \), \( \mathbf{a}_2 \) and \( \mathbf{a}_3 \).

\[
\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.
\]

**Solution.**

The question

Is \( \mathbf{b} \) a linear combination of \( \mathbf{a}_1 \), \( \mathbf{a}_2 \), and \( \mathbf{a}_3 \)?

is equivalent to the question

Does the vector equation \( x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 = \mathbf{b} \) have a solution?

The equation

\[
\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 5 \\ -6 \\ -8 \end{bmatrix} x_3 = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}
\]

\[
\begin{array}{ccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b}
\end{array}
\]

has the same solution set as the linear system whose augmented matrix is

\[
M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}
\]

Row reduce \( M \) until the pivot positions are visible:

\[
M \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

The linear system corresponding to \( M \) has a solution, so the vector equation (*) has a solution, and therefore \( \mathbf{b} \) is a linear combination of \( \mathbf{a}_1 \), \( \mathbf{a}_2 \), and \( \mathbf{a}_3 \).

---

11. (§1.3#13) Determine if \( \mathbf{b} \) is a linear combination of the vectors formed from the columns of the matrix \( \mathbf{A} \).

\[
\mathbf{A} = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}.
\]

**Solution.**

Denote the columns of \( \mathbf{A} \) by \( \mathbf{a}_1 \), \( \mathbf{a}_2 \), \( \mathbf{a}_3 \). To determine if \( \mathbf{b} \) is a linear combination of these columns, use the boxed fact in the subsection Linear Combinations. Row reduce the augmented matrix

\[
[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}] = \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

The system for this augmented matrix is inconsistent, so \( \mathbf{b} \) is not a linear combination of the columns of \( \mathbf{A} \).

---

12. (§1.3#16) Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} \), and \( \mathbf{y} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix} \). For what value(s) of \( h \) is \( \mathbf{y} \) in the plane generated by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \)?
13. (§1.3#24) Mark each statement True or False. Justify each answer.

(a) When \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors, \( \text{Span}\{\mathbf{u}, \mathbf{v}\} \) contains only the line through \( \mathbf{u} \) and the origin, and the line through \( \mathbf{v} \) and the origin.

(b) Any list of five real numbers is a vector in \( \mathbb{R}^5 \).

(c) Asking whether the linear system corresponding to an augmented matrix
\[
\begin{bmatrix}
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b}
\end{bmatrix}
\]
has a solution amounts to asking whether \( \mathbf{b} \) is in \( \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \).

(d) The vector \( \mathbf{v} \) results when a vector \( \mathbf{u} - \mathbf{v} \) is added to the vector \( \mathbf{v} \).

(e) The weights \( c_1, \ldots, c_p \) in a linear combination \( c_1 \mathbf{v} + \cdots + c_p \mathbf{v} \) cannot all be zero.

Solution.

a. False. \( \text{Span}\{\mathbf{u}, \mathbf{v}\} \) can be a plane.

b. True. See the beginning of the subsection Vectors in \( \mathbb{R}^n \).

c. True. See the comment following the definition of \( \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \).

d. False. \( (\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u} - \mathbf{v} + \mathbf{v} = \mathbf{u} \).

e. False. Setting all the weights equal to zero results in a legitimate linear combination of a set of vectors.