No books, notes, calculators, computers or cellphones.

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Rules for the exam:

1. Bring a photo ID for the inspection of the invigilator.
2. During the exam, people may be relocated with no reasons.
3. From five minutes before the end of the exam, you cannot hand in your exam any more and should wait in your seat until the end of the exam.
4. When the invigilator says that the exam is over, you should stop writing and remain seated. Please pass your exam to the nearest aisle.
5. Do not discuss before you leave the room, since your neighbor may change her/his solutions after hearing your conversation.
6. You are not allowed to leave until the invigilator has collected all exams and says that you can leave.
1. (a) Let \( T \) be the composition of linear transformations on \( \mathbb{R}^2 \) that first rotate counterclockwise about the origin by 90 degree, and then reflect about the line \( x_1 + x_2 = 0 \). Find its matrix.

**Solution.**

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
\]

The matrix is \( \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \).

**Solution #2.** The rotation has matrix \( B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). The reflection has matrix \( A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \). The matrix of \( T \) is

\[
AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

(b) Determine if \( A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \) is invertible? If not, explain why. If so, compute its inverse.

**Solution.** It can be found by row operations \([A|I] \sim [I|A^{-1}]\),

\[
[A|I] = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 \end{bmatrix} + R_1 \sim \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 \\ 0 & 3 & 4 & 3 & 0 \end{bmatrix} - R_3
\]

\[
\sim \begin{bmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 \\ 0 & 3 & 4 & 3 & 1 \end{bmatrix} + 3R_1
\]

\[
\sim \begin{bmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 \\ 0 & 3 & 4 & 3 & 1 \end{bmatrix} + R_2
\]

\[
\sim \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & -5 & 4 \\ 0 & 0 & 1 & -3 & 3 \end{bmatrix} \times (-1)
\]

\[
\sim \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 5 & -4 \\ 0 & 0 & 1 & -3 & 3 \end{bmatrix} \times (-1)
\]

Since we are able to row reduce \([A|I]\) to the form \([I|C]\), \( A \) is invertible and

\[
A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 5 & -4 & 3 \\ -3 & 3 & -2 \end{bmatrix}.
\]
2. Indicate if each of the following is a linear subspace. Explain why if it is not. (In this case, a correct answer with a wrong reason receives zero mark.) No justification needed if it is.

(a) All the vectors \[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]
with \( x_1 + x_3 = 1 \).

\textit{Solution.} No. The zero vector does not belong to the above set!

(b) The intersection of two subspaces of \( \mathbb{R}^n \).

\textit{Solution.} Yes

(c) The union of the \( x_1 \)-axis and the \( x_2 \)-axis on \( \mathbb{R}^2 \).

\textit{Solution.} No. The vectors \( e_1 \) and \( e_2 \) both belong to this set but \( e_1 + e_2 \) does not.
(6pt) 3. Let \( A = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n] \) be an \( n \times n \) matrix with column vectors \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \). For each of the following statements say if they are true or false and justify your answer. A correct answer with a wrong reason receives zero mark.

(a) If \( \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{n-1}\} \) is linearly independent and \( \det(A) = 0 \), then \( \vec{a}_n \) is a linear combination of \( \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{n-1}\} \).

\textit{Solution.} True. If not, then \( \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \) is linearly independent, and hence \( \det(A) \neq 0 \).

(b) There exists a vector \( \vec{c} \) which is not in the span of \( \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{n-1}\} \).

\textit{Solution.} True, since at least \( n \) vectors are required to span \( \mathbb{R}^n \).

(c) If \( \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{n-1}\} \) is linearly independent then there exists a vector \( \vec{c} \) such that the determinant of the matrix \( C = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{c}] \) obtained from replacing the last column by \( \vec{c} \) has nonzero determinant.

\textit{Solution.} True. By part (b), there is a vector \( \vec{c} \) which is not a linear combination of \( \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{n-1}\} \). Then by part (a), \( \det(A) \neq 0 \).
4. Let \( A = [\vec{a}_1 \, \vec{a}_2 \, \vec{a}_3 \, \vec{a}_4 \, \vec{a}_5] \) be the 4 \( \times \) 5 matrix whose columns are the vectors \( \vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \) and \( \vec{a}_5 \) in \( \mathbb{R}^4 \). Suppose the reduced echelon form of \( A \) is

\[
\text{REF}(A) = \begin{bmatrix} 1 & 0 & -2 & -4 & 0 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

(a) What is the rank of \( A \)? Explain briefly.

**Solution.** 3, because there are 3 pivot positions.

(b) What is the dimension of the nullspace of \( A \)? Explain briefly.

**Solution.** 2, because \( 5 - 3 = 2 \), by Rank Theorem.

Alternatively, it is 2 because there are two free variables \( x_3, x_4 \).

(c) Give a basis \( B \) for the nullspace space of \( A \).

**Solution.** \( x_3, x_4 \) free,

\[
\vec{x} = \begin{bmatrix} 2x_3 + 4x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]

A basis is \( \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \).

(d) The vector \( \vec{x} = \begin{bmatrix} 0 \\ 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \) is in the null space of \( A \). Find its coordinate vector \( [\vec{x}]_B \) in the basis \( B \) you give in part (c).

**Solution.** \( \vec{x} = -2\vec{v}_1 + \vec{v}_2 \). Thus \( [\vec{x}]_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \).

(The answer depends on part (c).)
(4pt) 5. (a) Find the determinant of the matrix
\[ A = \begin{bmatrix} 7 & -1 & 2 & 7 \\ 6 & 0 & 0 & 1 \\ 5 & 2 & 1 & 7 \\ 4 & 0 & 0 & 0 \end{bmatrix}. \]

**Solution.** Expand first in the 4th row, then in the 2nd row,
\[
\det A = -4 \begin{vmatrix} -1 & 2 & 7 \\ 0 & 0 & 1 \\ 2 & 1 & 7 \end{vmatrix} = (-4)(-1) \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = (-4)(-1)(-5) = -20.
\]

(6pt) (b) Find a basis for the eigenspace of the matrix
\[ A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \]
corresponding to eigenvalue 4.

**Solution.**
\[
A - 4I = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[ x_1, x_2 \text{ basic variables, } x_3, x_4 \text{ free,} \]
\[
x_1 = 2x_4, \quad x_2 = 3x_4, \quad \bar{x} = \begin{bmatrix} 2x_4 \\ 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}
\]

A basis for the eigenspace is \[ \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}. \]
(6pt) 6. Circle the correct answers. No justifications necessary.

(a) True or false? For \( n \times n \) matrices \( A, B \) we have \( \det(A + B) = \det A + \det B \).

Solution. Counterexample: \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \). We have \( \det A = \det B = 0 \) while \( A + B = I_2 \) and \( \det I_2 = 1 \).

(b) True or false? It is possible that a \( 5 \times 7 \) matrix has rank 6.

Solution. The rank is the dimension of the column space. The column space is a subspace of the target space \( \mathbb{R}^5 \), and cannot have dimension greater than 5.

(c) True or false? If \( A, B, C \) are square matrices, \( AB = C \), and \( A \) is invertible, then \( B = CA^{-1} \).

Solution. \( B = IB = A^{-1}AB = A^{-1}C \). Counterexample: \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \)

\[
C = AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad CA^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \neq B.
\]

(d) True or false? For a \( 3 \times 3 \) matrix \( A \) and a scalar \( r \) we have \( \det(rA) = r \det(A) \).

Solution. \( \det(rA) = r^3 \det(A) \).

(e) True or false? A square matrix with entirely zeros in the diagonal is never invertible.

Solution. Counterexample: \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

(f) True or false? Every spanning subset of \( \mathbb{R}^4 \) contains a basis for \( \mathbb{R}^4 \).

Solution. Call this spanning subset \( S \). Any maximally linearly independent subset \( E \) of \( S \) spans the span of \( S \). This subset \( E \) is a basis for \( \mathbb{R}^4 \).