Practice 2

based on an earlier version of Lay’s book.

1.5–1.9, skip 1.6

1.5: 6, 12, 24, 26, 32
1.7: 6, 14, 22, 26, 36
1.8: 4, 12, 22, 26, 30, 34
1.9: 8, 12, 14, 18, 24
1.5 Exercises

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1. \begin{align*}
2x_1 - 5x_2 + 8x_3 &= 0 \\
-2x_1 + 7x_2 + x_3 &= 0 \\
4x_1 + 2x_2 + 7x_3 &= 0
\end{align*}

2. \begin{align*}
-x_1 - 3x_2 + 7x_3 &= 0 \\
-2x_1 + x_2 - 4x_3 &= 0 \\
x_1 + 2x_2 + 9x_3 &= 0
\end{align*}

3. \begin{align*}
-3x_1 + 5x_2 - 7x_3 &= 0 \\
-6x_1 + 7x_2 + x_3 &= 0
\end{align*}

4. \begin{align*}
-x_1 + 7x_2 + 9x_3 &= 0 \\
x_1 - 2x_2 + 6x_3 &= 0
\end{align*}

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

5. \begin{align*}
x_1 + 3x_2 + x_3 &= 0 \\
-4x_1 - 9x_2 + 2x_3 &= 0 \\
-3x_2 - 6x_3 &= 0
\end{align*}

6. \begin{align*}
x_1 + 3x_2 - 5x_3 &= 0 \\
x_1 + 4x_2 - 8x_3 &= 0 \\
-3x_1 - 7x_2 + 9x_3 &= 0
\end{align*}

In Exercises 7–12, describe all solutions of \( Ax = 0 \) in parametric vector form, where \( A \) is row equivalent to the given matrix.

7. \begin{equation}
\begin{bmatrix}
1 & 3 & -3 & 7 \\
0 & 1 & -4 & 5
\end{bmatrix}
\end{equation}

8. \begin{equation}
\begin{bmatrix}
1 & -2 & -9 & 5 \\
0 & 1 & 2 & -6
\end{bmatrix}
\end{equation}

9. \begin{equation}
\begin{bmatrix}
3 & -9 & 6 \\
-1 & 3 & -2
\end{bmatrix}
\end{equation}

10. \begin{equation}
\begin{bmatrix}
1 & -4 & 2 & 0 & 3 & -5 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}

11. \begin{equation}
\begin{bmatrix}
1 & 5 & 2 & -6 & 9 & 0 \\
0 & 0 & 1 & -7 & 4 & -8 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}

12. \begin{equation}
\begin{bmatrix}
1 & 3 & 0 & -4 \\
2 & 6 & 0 & -8
\end{bmatrix}
\end{equation}

13. Suppose the solution set of a certain system of linear equations can be described as \( x_1 = 5 + 4x_3, x_2 = -2 - 7x_3 \), with \( x_3 \) free. Use vectors to describe this set as a line in \( \mathbb{R}^3 \).

14. Suppose the solution set of a certain system of linear equations can be described as \( x_1 = 3x_4, x_2 = 8 + x_4, x_3 = 2 - 5x_4 \), with \( x_4 \) free. Use vectors to describe this set as a “line” in \( \mathbb{R}^4 \).

15. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

\begin{align*}
x_1 + 3x_2 + x_3 &= 1 \\
-4x_1 - 9x_2 + 2x_3 &= -1 \\
-3x_2 - 6x_3 &= -3
\end{align*}

16. As in Exercise 15, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

\begin{align*}
x_1 + 3x_2 - 5x_3 &= 4 \\
x_1 + 4x_2 - 8x_3 &= 7 \\
-3x_1 - 7x_2 + 9x_3 &= -6
\end{align*}

17. Describe and compare the solution sets of \( x_1 + 9x_2 - 4x_3 = 0 \) and \( x_1 + 9x_2 - 4x_3 = -2 \).

18. Describe and compare the solution sets of \( x_1 - 3x_2 + 5x_3 = 0 \) and \( x_1 - 3x_2 + 5x_3 = 4 \).

In Exercises 19 and 20, find the parametric equation of the line through \( a \) parallel to \( b \).

19. \( a = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, b = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} \)

20. \( a = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}, b = \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix} \)

In Exercises 21 and 22, find a parametric equation of the line \( M \) through \( p \) and \( q \). [Hint: \( M \) is parallel to the vector \( q - p \). See the figure below.]

21. \( p = \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}, q = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} \)

22. \( p = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \)

The line through \( p \) and \( q \).

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A homogeneous equation is always consistent.

b. The equation \( Ax = 0 \) gives an explicit description of its solution set.
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c. The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable.

d. The equation $x = p + rv$ describes a line through $v$ parallel to $p$.

e. The solution set of $Ax = b$ is the set of all vectors of the form $w = p + vb$, where $vb$ is any solution of the equation $Ax = 0$.

24. a. If $x$ is a nontrivial solution of $Ax = 0$, then every entry in $x$ is nonzero.

b. The equation $x = x_2u + x_3v$, with $x_2$ and $x_3$ free (and neither $u$ nor $v$ a multiple of the other), describes a plane through the origin.

c. The equation $Ax = b$ is homogeneous if the zero vector is a solution.

d. The effect of adding $p$ to a vector is to move the vector in a direction parallel to $p$.

e. The solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

25. Prove Theorem 6:

a. Suppose $p$ is a solution of $Ax = b$, so that $Ap = b$. Let $vb$ be any solution of the homogeneous equation $Ax = 0$, and let $w = p + vb$. Show that $w$ is a solution of $Ax = b$.

b. Let $w$ be any solution of $Ax = b$, and define $vb = w - p$. Show that $vb$ is a solution of $Ax = 0$. This shows that every solution of $Ax = b$ has the form $w = p + vb$, with $p$ a particular solution of $Ax = b$ and $vb$ a solution of $Ax = 0$.

26. Suppose $Ax = b$ has a solution. Explain why the solution is unique precisely when $Ax = 0$ has only the trivial solution.

27. Suppose $A$ is the $3 \times 3$ zero matrix (with all zero entries). Describe the solution set of the equation $Ax = 0$.

28. If $b \neq 0$, can the solution set of $Ax = b$ be a plane through the origin? Explain.

In Exercises 29–32, (a) does the equation $Ax = 0$ have a nontrivial solution and (b) does the equation $Ax = b$ have at least one solution for every possible $b$?

29. $A$ is a $3 \times 3$ matrix with three pivot positions.

30. $A$ is a $3 \times 3$ matrix with two pivot positions.

31. $A$ is a $3 \times 2$ matrix with two pivot positions.

32. $A$ is a $2 \times 4$ matrix with two pivot positions.

33. Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$, find one nontrivial solution of $Ax = 0$ by inspection. [Hint: Think of the equation $Ax = 0$ written as a vector equation.]

34. Given $A = \begin{bmatrix} -4 & -6 \\ -8 & 12 \\ 6 & -9 \end{bmatrix}$, find one nontrivial solution of $Ax = 0$ by inspection.

35. Construct a $3 \times 3$ nonzero matrix $A$ such that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

36. Construct a $3 \times 3$ nonzero matrix $A$ such that the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

37. Construct a $2 \times 2$ matrix $A$ such that the solution set of the equation $Ax = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $b$ in $\mathbb{R}^2$ such that the solution set of $Ax = b$ is not a line in $\mathbb{R}^2$ parallel to the solution set of $Ax = 0$. Why does this not contradict Theorem 6?

38. Suppose $A$ is a $3 \times 3$ matrix and $y$ is a vector in $\mathbb{R}^3$ such that the equation $Ax = y$ does not have a solution. Does there exist a vector $z$ in $\mathbb{R}^3$ such that the equation $Ax = z$ has a unique solution? Discuss.

39. Let $A$ be an $m \times n$ matrix and let $u$ be a vector in $\mathbb{R}^n$ that satisfies the equation $Ax = 0$. Show that for any scalar $c$, the vector $cu$ also satisfies $Ax = 0$. [That is, show that $A(cu) = 0$.]

40. Let $A$ be an $m \times n$ matrix, and let $u$ and $v$ be vectors in $\mathbb{R}^n$ with the property that $Au = 0$ and $Av = 0$. Explain why $A(u + v)$ must be the zero vector. Then explain why $A(cu + dv) = 0$ for each pair of scalars $c$ and $d$.

Solutions to Practice Problems

1. Row reduce the augmented matrix:

$$
\begin{bmatrix}
1 & 4 & -5 & 0 \\
2 & -1 & 8 & 9
\end{bmatrix}
~ \begin{bmatrix}
1 & 4 & -5 & 0 \\
0 & -9 & 18 & 9
\end{bmatrix}
~ \begin{bmatrix}
1 & 0 & 3 & 4 \\
0 & 1 & -2 & -1
\end{bmatrix}
$$
1.7 Exercises

In Exercises 1–4, determine if the vectors are linearly independent. Justify each answer.

1. \[
\begin{bmatrix}
5 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
7 \\
2 \\
-3
\end{bmatrix}, \begin{bmatrix}
9 \\
4 \\
-6
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
5 \\
-8
\end{bmatrix}, \begin{bmatrix}
-3 \\
4 \\
1
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
1 \\
-6 \\
9
\end{bmatrix}, \begin{bmatrix}
-3 \\
-8 \\
9
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
1 \\
-1 \\
4
\end{bmatrix}, \begin{bmatrix}
-3 \\
4 \\
-2
\end{bmatrix}
\]

In Exercises 5–8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

5. \[
\begin{bmatrix}
0 & 3 & -4 \\
3 & -7 & 1 \\
1 & 5 & -3
\end{bmatrix}, \begin{bmatrix}
8 & 4 \\
4 & 5 \\
2 & 1
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
-4 & 0 \\
0 & 1 \\
5 & 4
\end{bmatrix}, \begin{bmatrix}
-3 & 0 \\
1 & 3 \\
1 & 6
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
1 & -2 & -4 \\
4 & 5 & 5 \\
-3 & 7 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 3 \\
2 & 1 \\
1 & 4
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 3 & 3 & 2 \\
-3 & 7 & 1 & 2 \\
0 & 1 & 4 & 3
\end{bmatrix}
\]

In Exercises 9 and 10, (a) for what values of \( h \) is \( v_1 \) in Span \( \{v_1, v_2\} \), and (b) for what values of \( h \) is \( \{v_1, v_2, v_3\} \) linearly dependent? Justify each answer.

9. \( v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix} \)

10. \( v_1 = \begin{bmatrix} -5 \\ -3 \\ -9 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 10 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -2 \\ h \end{bmatrix} \)

In Exercises 11–14, find the value(s) of \( h \) for which the vectors are linearly dependent. Justify each answer.

11. \[
\begin{bmatrix}
1 \\
4 \\
1
\end{bmatrix}, \begin{bmatrix}
3 \\
7 \\
5
\end{bmatrix}, \begin{bmatrix}
-1 \\
-7 \\
h
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
-6 \\
-3 \\
4
\end{bmatrix}, \begin{bmatrix}
8 \\
h \\
4
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
1 \\
5 \\
-3
\end{bmatrix}, \begin{bmatrix}
-2 \\
-9 \\
6
\end{bmatrix}, \begin{bmatrix}
3 \\
h \\
-9
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
-5 \\
-7 \\
-3
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
h
\end{bmatrix}
\]

Determine by inspection whether the vectors in Exercises 15–20 are linearly independent. Justify each answer.

15. \[
\begin{bmatrix}
5 \\
1 \\
2
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
-1
\end{bmatrix}, \begin{bmatrix}
1 \\
7 \\
-1
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
4 \\
-2 \\
2
\end{bmatrix}, \begin{bmatrix}
6 \\
-3 \\
9
\end{bmatrix}
\]

17. \[
\begin{bmatrix}
3 \\
-1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
5
\end{bmatrix}, \begin{bmatrix}
-6 \\
5 \\
4
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
4 \\
1 \\
-1
\end{bmatrix}, \begin{bmatrix}
-3 \\
5 \\
2
\end{bmatrix}, \begin{bmatrix}
8 \\
1 \\
1
\end{bmatrix}
\]

19. \[
\begin{bmatrix}
-8 \\
12 \\
-4
\end{bmatrix}, \begin{bmatrix}
2 \\
-3 \\
-1
\end{bmatrix}
\]

20. \[
\begin{bmatrix}
1 \\
4 \\
1
\end{bmatrix}, \begin{bmatrix}
-2 \\
5 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

21. a. The columns of a matrix \( A \) are linearly independent if the equation \( Ax = 0 \) has the trivial solution.
   b. If \( S \) is a linearly dependent set, then each vector is a linear combination of the other vectors in \( S \).
   c. The columns of any \( 4 \times 5 \) matrix are linearly dependent.
   d. If \( x \) and \( y \) are linearly independent, and if \( \{x, y, z\} \) is linearly dependent, then \( z \) is in Span \( \{x, y\} \).

22. a. Two vectors are linearly dependent if and only if they lie on a line through the origin.
   b. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.
   c. If \( x \) and \( y \) are linearly independent, and if \( z \) is in Span \( \{x, y\} \), then \( \{x, y, z\} \) is linearly dependent.
   d. If a set in \( \mathbb{R}^n \) is linearly dependent, then the set contains more vectors than there are entries in each vector.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

23. \( A \) is a \( 3 \times 3 \) matrix with linearly independent columns.

24. \( A \) is a \( 2 \times 2 \) matrix with linearly dependent columns.

25. \( A \) is a \( 4 \times 2 \) matrix, \( A = [a_1, a_2] \), and \( a_2 \) is not a multiple of \( a_1 \).

26. \( A \) is a \( 4 \times 3 \) matrix, \( A = [a_1, a_2, a_3] \), such that \( \{a_1, a_2\} \) is linearly independent and \( a_3 \) is not in Span \( \{a_1, a_2\} \).

27. How many pivot columns must a \( 7 \times 5 \) matrix have if its columns are linearly independent? Why?

28. How many pivot columns must a \( 5 \times 7 \) matrix have if its columns span \( \mathbb{R}^5 \)? Why?

29. Construct \( 3 \times 2 \) matrices \( A \) and \( B \) such that \( Ax = 0 \) has only the trivial solution and \( Bx = 0 \) has a nontrivial solution.
30. a. Fill in the blank in the following statement: “If \( A \) is an \( m \times n \) matrix, then the columns of \( A \) are linearly independent if and only if \( A \) has _____ pivot columns.”

b. Explain why the statement in (a) is true.

Exercises 31 and 32 should be solved without performing row operations. [Hint: Write \( Ax = 0 \) as a vector equation.]

31. Given \( A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \end{bmatrix} \), observe that the third column is the sum of the first two columns. Find a nontrivial solution of \( Ax = 0 \).

32. Given \( A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix} \), observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of \( Ax = 0 \).

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a counterexample to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

33. If \( v_1, \ldots, v_4 \) are in \( \mathbb{R}^3 \) and \( v_3 = 2v_1 + v_2 \), then \( \{v_1, v_2, v_3, v_4\} \) is linearly dependent.

34. If \( v_1, \ldots, v_4 \) are in \( \mathbb{R}^3 \) and \( v_3 = 0 \), then \( \{v_1, v_2, v_3, v_4\} \) is linearly dependent.

35. If \( v_1 \) and \( v_2 \) are in \( \mathbb{R}^3 \) and \( v_2 \) is not a scalar multiple of \( v_1 \), then \( \{v_1, v_2\} \) is linearly independent.

36. If \( v_1, \ldots, v_4 \) are in \( \mathbb{R}^4 \) and \( v_3 \) is not a linear combination of \( v_1, v_2, v_4 \), then \( \{v_1, v_2, v_3, v_4\} \) is linearly independent.

37. If \( v_1, \ldots, v_4 \) are in \( \mathbb{R}^3 \) and \( \{v_1, v_2, v_3\} \) is linearly dependent, then \( \{v_1, v_2, v_3, v_4\} \) is also linearly dependent.

38. If \( v_1, \ldots, v_4 \) are linearly independent vectors in \( \mathbb{R}^4 \), then \( \{v_1, v_2, v_3\} \) is also linearly independent. [Hint: Think about \( x_1v_1 + x_2v_2 + x_3v_3 + 0v_4 = 0 \).]

39. Suppose \( A \) is an \( m \times n \) matrix with the property that for all \( b \) in \( \mathbb{R}^m \) the equation \( Ax = b \) has at most one solution. Use the definition of linear independence to explain why the columns of \( A \) must be linearly independent.

40. Suppose an \( m \times n \) matrix \( A \) has \( n \) pivot columns. Explain why for each \( b \) in \( \mathbb{R}^m \) the equation \( Ax = b \) has at most one solution. [Hint: Explain why \( Ax = b \) cannot have infinitely many solutions.]

[M] In Exercises 41 and 42, use as many columns of \( A \) as possible to construct a matrix \( B \) with the property that the equation \( Bx = 0 \) has only the trivial solution. Solve \( Bx = 0 \) to verify your work.

41. \( A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix} \)

42. \( A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & 5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix} \)

43. [M] With \( A \) and \( B \) as in Exercise 41, select a column \( v \) of \( A \) that was not used in the construction of \( B \) and determine if \( v \) is in the set spanned by the columns of \( B \). (Describe your calculations.)

44. [M] Repeat Exercise 43 with the matrices \( A \) and \( B \) from Exercise 42. Then give an explanation for what you discover, assuming that \( B \) was constructed as specified.
1.8 Introduction to Linear Transformations

The final example is not geometrical; instead, it shows how a linear mapping can transform one type of data into another.

**EXAMPLE 6** A company manufactures two products, B and C. Using data from Example 7 in Section 1.3, we construct a “unit cost” matrix, \( U = \begin{bmatrix} b & c \end{bmatrix} \), whose columns describe the “costs per dollar of output” for the products:

\[
\begin{array}{c|cc}
\text{Product} & \text{B} & \text{C} \\
\hline
\text{Materials} & .45 & .40 \\
\text{Labor} & .25 & .35 \\
\text{Overhead} & .15 & .15 \\
\end{array}
\]

Let \( x = (x_1, x_2) \) be a “production” vector, corresponding to \( x_1 \) dollars of product B and \( x_2 \) dollars of product C, and define \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) by

\[
T(x) = Ux = x_1 \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} + x_2 \begin{bmatrix} .40 \\ .35 \\ .15 \end{bmatrix} = \begin{bmatrix} \text{Total cost of materials} \\ \text{Total cost of labor} \\ \text{Total cost of overhead} \end{bmatrix}
\]

The mapping \( T \) transforms a list of production quantities (measured in dollars) into a list of total costs. The linearity of this mapping is reflected in two ways:

1. If production is increased by a factor of, say, 4, from \( x \) to \( 4x \), then the costs will increase by the same factor, from \( T(x) \) to \( 4T(x) \).
2. If \( x \) and \( y \) are production vectors, then the total cost vector associated with the combined production \( x + y \) is precisely the sum of the cost vectors \( T(x) \) and \( T(y) \).

**Practice Problems**

1. Suppose \( T : \mathbb{R}^5 \to \mathbb{R}^2 \) and \( T(x) = Ax \) for some matrix \( A \) and for each \( x \) in \( \mathbb{R}^5 \). How many rows and columns does \( A \) have?

2. Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). Give a geometric description of the transformation \( x \mapsto Ax \).

3. The line segment from \( \mathbf{0} \) to a vector \( \mathbf{u} \) is the set of points of the form \( t\mathbf{u} \), where \( 0 \leq t \leq 1 \). Show that a linear transformation \( T \) maps this segment into the segment between \( \mathbf{0} \) and \( T(\mathbf{u}) \).

**1.8 Exercises**

1. Let \( A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \), and define \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T(x) = Ax \). Find the images under \( T \) of \( \mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \).

2. Let \( A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \), \( \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \), and \( \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \). Define \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) by \( T(x) = Ax \). Find \( T(\mathbf{u}) \) and \( T(\mathbf{v}) \).
In Exercises 3–6, with $T$ defined by $T(x) = Ax$, find a vector $x$ whose image under $T$ is $b$, and determine whether $x$ is unique.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ 1 \\ -6 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

7. Let $A$ be a $6 \times 5$ matrix. What must $a$ and $b$ be in order to define $T : \mathbb{R}^a \rightarrow \mathbb{R}^b$ by $T(x) = Ax$?

8. How many rows and columns must a matrix $A$ have in order to define a mapping from $\mathbb{R}^a$ into $\mathbb{R}^b$ by the rule $T(x) = Ax$?

For Exercises 9 and 10, find all $x$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $x \mapsto Ax$ for the given matrix $A$.

9. $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

11. Let $b = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$, and let $A$ be the matrix in Exercise 9. Is $b$ in the range of the linear transformation $x \mapsto Ax$? Why or why not?

12. Let $b = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, and let $A$ be the matrix in Exercise 10. Is $b$ in the range of the linear transformation $x \mapsto Ax$? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation $T$. (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what $T$ does to each vector $x$ in $\mathbb{R}^2$.

13. $T(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x$

14. $T(x) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} x$

15. $T(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$

16. $T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$

17. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that $T$ is linear to find the images under $T$ of $3u$, $2v$, and $3u + 2v$.

18. The figure shows vectors $u$, $v$, and $w$, along with the images $T(u)$ and $T(v)$ under the action of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Copy this figure carefully, and draw the image $T(w)$ as accurately as possible. [Hint: First, write $w$ as a linear combination of $u$ and $v$.]

19. Let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $y_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $y_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps $e_1$ into $y_1$ and maps $e_2$ into $y_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

20. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $v_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps $x$ into $x_1v_1 + x_2v_2$. Find a matrix $A$ such that $T(x) = Ax$ for each $x$.

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A linear transformation is a special type of function.
   b. If $A$ is a $3 \times 5$ matrix and $T$ is a transformation defined by $T(x) = Ax$, then the domain of $T$ is $\mathbb{R}^3$.
   c. If $A$ is an $m \times n$ matrix, then the range of the transformation $x \mapsto Ax$ is $\mathbb{R}^m$.
   d. Every linear transformation is a matrix transformation.
22. Every matrix transformation is a linear transformation.

b. The codomain of the transformation \( x \mapsto Ax \) is the set of all linear combinations of the columns of \( A \).

c. If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation and if \( c \in \mathbb{R}^m \), then a uniqueness question is “Is \( c \) in the range of \( T \)?”

d. A linear transformation preserves the operations of vector addition and scalar multiplication.

e. The superposition principle is a physical description of a linear transformation.

23. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the linear transformation that reflects each point through the \( x_1 \)-axis. (See Practice Problem 2.) Make two sketches similar to Fig. 6 that illustrate properties (i) and (ii) of a linear transformation.

24. Suppose vectors \( v_1, \ldots, v_p \) span \( \mathbb{R}^n \), and let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear transformation. Suppose \( T(v_i) = 0 \) for \( i = 1, \ldots, p \). Show that \( T \) is the zero transformation. That is, show that if \( x \) is any vector in \( \mathbb{R}^n \), then \( T(x) = 0 \).

25. Given \( v \neq 0 \) and \( p \) in \( \mathbb{R}^n \), the line through \( p \) in the direction of \( v \) has the parametric equation \( x = p + tv \). Show that a linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) maps this line onto another line or onto a single point (a degenerate line).

26. Let \( u \) and \( v \) be linearly independent vectors in \( \mathbb{R}^3 \), and let \( P \) be the plane through \( u, v, \) and \( 0 \). The parametric equation of \( P \) is \( x = su + tv \) (with \( s, t \) in \( \mathbb{R} \)). Show that a linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) maps \( P \) onto a plane through \( 0 \), or onto a line through \( 0 \), or onto just the origin in \( \mathbb{R}^3 \). What must be true about \( T(u) \) and \( T(v) \) in order for the image of the plane \( P \) to be a plane?

27. a. Show that the line segment from \( p \) to \( q \) is the set of points of the form \( (1 - t)p + tq \) for \( 0 \leq t \leq 1 \) (as shown in the figure below). Show that a linear transformation \( T \) maps this line segment onto a line segment or onto a single point.

b. The line segment from \( p \) to \( q \) is the set of points of the form \( (1 - t)p + tq \) for \( 0 \leq t \leq 1 \) (as shown in the figure below). Show that a linear transformation \( T \) maps this line segment onto a line segment or onto a single point.

28. Let \( u \) and \( v \) be vectors in \( \mathbb{R}^n \). It can be shown that the set \( P \) of all points in the parallelogram determined by \( u \) and \( v \) has the form \( au + bv \), for \( 0 \leq a \leq 1, 0 \leq b \leq 1 \). Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Explain why the image of a point in \( P \) under the transformation \( T \) lies in the parallelogram determined by \( T(u) \) and \( T(v) \).

29. Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by \( f(x) = mx + b \).

a. Show that \( f \) is a linear transformation when \( b = 0 \).

b. Find a property of a linear transformation that is violated when \( b \neq 0 \).

c. Why is \( f \) called a linear function?

30. An affine transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) has the form \( T(x) = Ax + b \), with \( A \) an \( m \times n \) matrix and \( b \) in \( \mathbb{R}^m \). Show that \( T \) is not a linear transformation when \( b \neq 0 \). (Affine transformations are important in computer graphics.)

31. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation, and let \( \{v_1, v_2, v_3\} \) be a linearly dependent set in \( \mathbb{R}^n \). Explain why the set \( \{T(v_1), T(v_2), T(v_3)\} \) is linearly dependent.

In Exercises 32–36, column vectors are written as rows, such as \( x = (x_1, x_2) \), and \( T(x) \) is written as \( T(x_1, x_2) \).

32. Show that the transformation \( T \) defined by \( T(x_1, x_2) = (4x_1 - 2x_2, 3x_1) \) is not linear.

33. Show that the transformation \( T \) defined by \( T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2) \) is not linear.

34. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Show that if \( T \) maps two linearly independent vectors onto a linearly dependent set, then the equation \( T(x) = 0 \) has a nontrivial solution. [Hint: Suppose \( u \) and \( v \) in \( \mathbb{R}^n \) are linearly independent and yet \( T(u) \) and \( T(v) \) are linearly dependent. Then \( c_1 T(u) + c_2 T(v) = 0 \) for some weights \( c_1 \) and \( c_2 \), not both zero. Use this equation.]

35. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the transformation that reflects each vector \( x = (x_1, x_2, x_3) \) through the plane \( x_3 = 0 \) onto \( T(x) = (x_1, x_2, -x_3) \). Show that \( T \) is a linear transformation. [See Example 4 for ideas.]

36. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the transformation that projects each vector \( x = (x_1, x_2, x_3) \) onto the plane \( x_2 = 0 \), so \( T(x) = (x_1, 0, x_3) \). Show that \( T \) is a linear transformation.

[M] In Exercises 37 and 38, the given matrix determines a linear transformation \( T \). Find all \( x \) such that \( T(x) = 0 \).

\[
\begin{bmatrix}
4 & -2 & 5 & -5 \\
-9 & 7 & -8 & 0 \\
-6 & 4 & 5 & 3 \\
5 & -3 & 8 & -4
\end{bmatrix}
\]

37.

\[
\begin{bmatrix}
4 & -2 & 5 & -5 \\
-9 & 7 & -8 & 0 \\
-6 & 4 & 5 & 3 \\
5 & -3 & 8 & -4
\end{bmatrix}
\]

38.

\[
\begin{bmatrix}
-9 & -4 & -9 & 4 \\
5 & -8 & -7 & 6 \\
7 & 11 & 16 & -9 \\
9 & -7 & -4 & 5
\end{bmatrix}
\]
39. [M] Let \( b = \begin{bmatrix} 7 \\ 5 \\ 9 \\ 7 \end{bmatrix} \) and let \( A \) be the matrix in Exercise 37. Is \( b \) in the range of the transformation \( x \mapsto Ax \)? If so, find an \( x \) whose image under the transformation is \( b \).

40. [M] Let \( b = \begin{bmatrix} -7 \\ -13 \\ -5 \end{bmatrix} \) and let \( A \) be the matrix in Exercise 38. Is \( b \) in the range of the transformation \( x \mapsto Ax \)? If so, find an \( x \) whose image under the transformation is \( b \).

### Solution to Practice Problems

1. \( A \) must have five columns for \( Ax \) to be defined. \( A \) must have two rows for the codomain of \( T \) to be \( \mathbb{R}^2 \).

2. Plot some random points (vectors) on graph paper to see what happens. A point such as \((4, 1)\) maps into \((4, -1)\). The transformation \( x \mapsto Ax \) reflects points through the \( x \)-axis (or \( x_1 \)-axis).

3. Let \( x = tu \) for some \( t \) such that \( 0 \leq t \leq 1 \). Since \( T \) is linear, \( T(tu) = tT(u) \), which is a point on the line segment between \( 0 \) and \( T(u) \).

### 1.9 The Matrix of a Linear Transformation

Whenever a linear transformation \( T \) arises geometrically or is described in words, we usually want a “formula” for \( T(x) \). The discussion that follows shows that every linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is actually a matrix transformation \( x \mapsto Ax \) and that important properties of \( T \) are intimately related to familiar properties of \( A \). The key to finding \( A \) is to observe that \( T \) is completely determined by what it does to the columns of the \( n \times n \) identity matrix \( I_n \).

**Example 1**

The columns of \( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) are \( e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Suppose \( T \) is a linear transformation from \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \) such that

\[
T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}
\]

With no additional information, find a formula for the image of an arbitrary \( x \) in \( \mathbb{R}^2 \).

**Solution**

Write

\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2
\]
1.9 Exercises

In Exercises 1–10, assume that $T$ is a linear transformation. Find the standard matrix of $T$.

1. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(e_1) = (3, 1, 3, 1)$ and $T(e_2) = (-5, 2, 0, 0)$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

2. $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T(e_1) = (1, 3)$, $T(e_2) = (4, -7)$, and $T(e_3) = (-5, 4)$, where $e_1$, $e_2$, $e_3$ are the columns of the $3 \times 3$ identity matrix.

3. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotates points (about the origin) through $3\pi/2$ radians (counterclockwise).

4. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotates points (about the origin) through $-\pi/4$ radians (clockwise). [Hint: $T(e_1) = (1/\sqrt{2}, -1/\sqrt{2})$.]

5. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a vertical shear transformation that maps $e_1$ into $e_1 - 2e_2$ but leaves the vector $e_2$ unchanged.

6. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a horizontal shear transformation that leaves $e_1$ unchanged and maps $e_2$ into $e_2 + 3e_1$.

7. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal $x_1$-axis. [Hint: $T(e_1) = (-1/\sqrt{2}, 1/\sqrt{2})$.]

8. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ first reflects points through the horizontal $x_1$-axis and then reflects points through the line $x_1 = x_2$.

9. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ first performs a horizontal shear that transforms $e_2$ into $e_2 - 2e_1$ (leaving $e_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.

10. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ first reflects points through the vertical $x_2$-axis and then rotates points $\pi/2$ radians.

11. A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ first reflects points through the $x_1$-axis and then reflects points through the $x_2$-axis. Show that $T$ can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?

12. Show that the transformation in Exercise 8 is merely a rotation about the origin. What is the angle of the rotation?

13. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation such that $T(e_1)$ and $T(e_2)$ are the vectors shown in the figure. Using the figure, sketch the vector $T(2, 1)$.

14. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation with standard matrix $A = [a_1 \ a_2]$, where $a_1$ and $a_2$ are shown in the figure. Using the figure, draw the image of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ under the transformation $T$.

In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

15. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

16. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$
In Exercises 17–20, show that \( T \) is a linear transformation by finding a matrix that implements the mapping. Note that \( x_1, x_2, \ldots \) are not vectors but are entries in vectors.

17. \( T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4) \)
18. \( T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2) \)
19. \( T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3) \)
20. \( T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4 \) \( (T : \mathbb{R}^4 \rightarrow \mathbb{R}) \)

21. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear transformation such that \( T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2) \). Find \( x \) such that \( T(x) = (3, 8) \).
22. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a linear transformation such that \( T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2) \). Find \( x \) such that \( T(x) = (-1, 4, 9) \).

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A linear transformation \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is completely determined by its effect on the columns of the \( n \times n \) identity matrix.
   b. If \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) rotates vectors about the origin through an angle \( \phi \), then \( T \) is a linear transformation.
   c. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.
   d. A mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is onto \( \mathbb{R}^m \) if every vector \( x \) in \( \mathbb{R}^m \) maps to some vector in \( \mathbb{R}^n \).
   e. If \( A \) is a \( 3 \times 2 \) matrix, then the transformation \( x \mapsto Ax \) cannot be one-to-one.

24. a. Not every linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is a matrix transformation.
   b. The columns of the standard matrix for a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) are the images of the columns of the \( n \times n \) identity matrix.
   c. The standard matrix of a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) that reflects points through the horizontal axis, the vertical axis, or the origin has the form \( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \), where \( a \) and \( d \) are \( \pm 1 \).
   d. A mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one-to-one if each vector in \( \mathbb{R}^n \) maps onto a unique vector in \( \mathbb{R}^m \).
   e. If \( A \) is a \( 3 \times 2 \) matrix, then the transformation \( x \mapsto Ax \) cannot map \( \mathbb{R}^2 \) onto \( \mathbb{R}^3 \).

In Exercises 25–28, determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify each answer.

25. The transformation in Exercise 17
26. The transformation in Exercise 2
27. The transformation in Exercise 19
28. The transformation in Exercise 14

In Exercises 29 and 30, describe the possible echelon forms of the standard matrix for a linear transformation \( T \). Use the notation of Example 1 in Section 1.2.

29. \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) is one-to-one.
30. \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) is onto.

31. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation, with \( A \) its standard matrix. Complete the following statement to make it true: “\( T \) is one-to-one if and only if \( A \) has ____ pivot columns.” Explain why the statement is true. [Hint: Look in the exercises for Section 1.7.]

32. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation, with \( A \) its standard matrix. Complete the following statement to make it true: “\( T \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^m \) if and only if \( A \) has ____ pivot columns.” Find some theorems that explain why the statement is true.

33. Verify the uniqueness of \( A \) in Theorem 10. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation such that \( T(x) = Bx \) for some \( n \times n \) matrix \( B \). Show that if \( A \) is the standard matrix for \( T \), then \( A = B \). [Hint: Show that \( A \) and \( B \) have the same columns.]

34. Why is the question “Is the linear transformation \( T \) onto?” an existence question?

35. If a linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^m \), can you give a relation between \( m \) and \( n \)? If \( T \) is one-to-one, what can you say about \( m \) and \( n \)?

36. Let \( S : \mathbb{R}^p \rightarrow \mathbb{R}^q \) and \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be linear transformations. Show that the mapping \( u \mapsto T(S(u)) \) is a linear transformation (from \( \mathbb{R}^p \) to \( \mathbb{R}^m \)). [Hint: Compute \( T(S(cu + dv)) \) for \( u, v \) in \( \mathbb{R}^p \) and scalars \( c \) and \( d \). Justify each step of the computation, and explain why this computation gives the desired conclusion.]

[M] In Exercises 37–40, let \( T \) be the linear transformation whose standard matrix is given. In Exercises 37 and 38, decide if \( T \) is a one-to-one mapping. In Exercises 39 and 40, decide if \( T \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^m \). Justify your answers.

37. \[
\begin{bmatrix}
-5 & 10 & -5 & 4 \\
8 & 3 & -4 & 7 \\
4 & -9 & 5 & -3 \\
-3 & -2 & 5 & 4 \\
\end{bmatrix}
\]
38. \[
\begin{bmatrix}
7 & 5 & 4 & -9 \\
10 & 6 & 16 & -4 \\
12 & 8 & 12 & 7 \\
-8 & -6 & -2 & 5 \\
\end{bmatrix}
\]