Midterm Exam 2

Math 221, Sect 202, March 29, 2007

Name (print): ____________________________

Student No.: ____________________________

There are six problems worth a total of 60 marks. No notes/calculator allowed.

1. [7/3pt] (a) Compute the determinant of the matrix
\[
A = \begin{bmatrix}
2 & 1 & -2 & 10 \\
3 & 2 & 2 & 1 \\
3 & 2 & 2 & 2 \\
5 & 4 & 3 & 4 \\
\end{bmatrix}.
\]

(b) Let \( E \) be the rectangular region in \( \mathbb{R}^4 \):
\[
0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 2, \quad 0 \leq x_3 \leq 3, \quad 0 \leq x_4 \leq 4.
\]

Find the volume of the image of \( E \) under the map \( x \mapsto Ax \).

(Part b is from §3.3 and is not covered.)
2. [10pt] Find bases for Nul $A$ and Col $A$, where $A = \begin{bmatrix} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 2 \end{bmatrix}$. 
3. [6/4pt] (This problem is not covered in MATH 221.)
Let \( f(t) = (1 + t)^2 \), \( f'(t) = 2(1 + t) \) its first derivative, and \( f''(t) = 2 \) its second derivative. They belong to \( \mathbb{P}_2 \), the vector space of polynomials with degree at most 2.
(a) Show that \( \mathcal{B} = \{ f, f', f'' \} \) is basis for \( \mathbb{P}_2 \).
(b) Find the coordinates of \( g(t) = 3t^2 + 2t + 1 \) with respect to the basis \( \mathcal{B} \).
4. [8pt] Let

\[
A = \begin{bmatrix}
-2 & 3 & -5 & 4 & 7 \\
6 & -9 & 15 & -12 & -21 \\
2 & -3 & 5 & -4 & -7 \\
-2 & 3 & -5 & 4 & 7
\end{bmatrix}
\]

Determine the dimension of the null space \( \text{Nul} \ A \) “by inspection” by appealing to the Rank Theorem. You only get half marks if you find it directly without using the Rank Theorem.
5. [7/3pt] (a) Find all eigenvalues of the matrix \( A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \). For each of them find a corresponding eigenvector.
(b) Find a \( 2 \times 2 \) diagonal matrix \( D \) and a \( 2 \times 2 \) invertible matrix \( P \) so that \( A = PDP^{-1} \).
(Part b is from §4.3 and is not covered in MT2.)
6. [12pt] Decide whether the following statements are true or false, and *justify* your answer.

(a) Let \( A \) be a \( 4 \times 5 \) matrix. If \( A\vec{x} = 0 \) has only zero solution \( \vec{x} = 0 \), then \( A \) is invertible.

(b) If \( A \) is a \( 3 \times 3 \) matrix, then \( \det(2A) = 8 \det A \).

(c) Each line in \( \mathbb{R}^n \) is a one-dimensional subspace of \( \mathbb{R}^n \).

(d) If the characteristic polynomial of a \( 4 \times 4 \) matrix \( A \) is \( (\lambda - 1)^2(\lambda - 2)(\lambda - 3) \), then the eigenspace of \( A \) corresponding to eigenvalue 1 has dimension 2.