Rules for the exam:

1. Bring a photo ID for the inspection of the invigilator.

2. During the exam, people may be relocated with no reasons.

3. From five minutes before the end of the exam, you cannot hand in your exam any more and should wait in your seat until the end of the exam.

4. When the invigilator says that the exam is over, you should stop writing and remain seated. Please pass your exam to the nearest aisle.

5. Do not discuss before you leave the room, since your neighbor may change her/his solutions after hearing your conversation.

6. You are not allowed to leave until the invigilator has collected all exams and says that you can leave.
1. Solve the following linear system and write the solution in parametric vector form.

\[
\begin{align*}
2x_1 + 4x_2 + 2x_3 + 2x_4 &= 6 \\
x_1 + 2x_2 + x_3 &= 3 \\
x_1 + x_2 - x_3 &= 3
\end{align*}
\]

**Solution.** Consider the augmented matrix

\[
\begin{bmatrix}
2 & 4 & 2 & 2 & 6 \\
1 & 2 & 1 & 0 & 3 \\
1 & 1 & -1 & 0 & 3
\end{bmatrix}
\]

R₁ ↔ R₂ \sim \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\
2 & 4 & 2 & 2 & 6 \\
1 & 1 & -1 & 0 & 3
\end{bmatrix} -2R₁

\sim \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\
0 & 0 & 0 & 2 & 0 \\
0 & -1 & -2 & 0 & 0
\end{bmatrix}

R₂ ↔ R₃

\sim \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\
0 & -1 & -2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0
\end{bmatrix} /2

\times(-1) \sim \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} -2R₂

\sim \begin{bmatrix} 1 & 0 & -3 & 0 & 3 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}

Now it’s in RREF.

No pivot in augmentation column: it’s consistent

No pivot in \(x₃\)-column, \(x₃\) is free.

\[
x₁ = 3 + 3x₃, \quad x₂ = -2x₃, \quad x₃\text{ free}, \quad x₄ = 0.
\]

\[
\vec{x} = \begin{bmatrix} 3 + 3x₃ \\
-2x₃ \\
x₃ \\
0
\end{bmatrix} = \begin{bmatrix} 3 \\
0 \\
0 \\
0
\end{bmatrix} + x₃ \begin{bmatrix} 3 \\
0 \\
0 \\
0
\end{bmatrix}
2. Write down all equations, if any, satisfied by the components $b_1, b_2, b_3$ of $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ that is in the Span of

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}.$$ 

Is the Span the entire $\mathbb{R}^3$? Why or why not?

**Solution.** $\vec{b}$ is in the Span if and only if the following augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & b_1 \\ 1 & 0 & 3 & b_2 \\ 0 & 1 & -2 & b_3 \end{bmatrix}$$

is consistent. It is equivalent to

$$\sim \begin{bmatrix} 1 & 1 & 0 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ 0 & 1 & -2 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \end{bmatrix}$$

This is in echelon form. To be consistent, the augmentation column cannot contain a pivot position, and hence

$$b_3 + b_2 - b_1 = 0.$$ 

The Span is not entire $\mathbb{R}^3$ since its components satisfy the above equation. For example, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ does not satisfy it and hence is not in the Span.
(5pt) 3. (a) Find all values of $h$, such that the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ h \\ 3 \\ 5 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

are linearly dependent.

**Solution.** We want the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ has a nontrivial solution $(c_1, c_2, c_3)$. We look at the augmented matrix,

$$\begin{bmatrix}
1 & 0 & -1 & 0 \\
1 & h & 0 & 0 \\
2 & 3 & 1 & 0 \\
3 & 5 & 2 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & h & 1 & 0 \\
0 & 3 & 3 & 0 \\
0 & 5 & 5 & 0
\end{bmatrix} \quad \text{R}_2 \leftrightarrow \text{R}_3$$

$$\sim \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 3 & 3 & 0 \\
0 & h & 1 & 0 \\
0 & 5 & 5 & 0
\end{bmatrix} \quad \text{R}_2/3 \sim \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & h & 1 & 0 \\
0 & 5 & 5 & 0
\end{bmatrix} \quad -5\text{R}_2$$

To to linearly dependent, we need a free variable (i.e., a non-pivot column).
Thus we need $1 - h = 0$, i.e., $h = 1$.

**Note.** (i) Since the augmentation column is $\vec{0}$, we can ignore it and only look at the coefficient matrix. (ii) Many students divided a row by $h$ or by $h - 1$. It is dangerous: If you divide by zero, you may lose a solution. For example, if you divide Row 3 of the last matrix by $h - 1$, $x_3$ is no longer a free variable.

(b) Choose a value of $h$ such that $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly dependent, and write down a linear dependence relation.

**Solution.** For $h = 1$, the RREF of our augmented matrix is

$$\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$

It implies $c_3$ is free and

$$c_1 = c_3, \quad c_2 = -c_3.$$

Taking for example $c_3 = 1$, we have

$$\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}.$$
4. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$
T \left( \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
$$

(a) What is $T \left( \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right)$?

**Solution.** By the properties of linear transformations,

$$
T \left( \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right) = T(4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = 4T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.
$$

(b) What is $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$?

**Hint.** Write $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination.

**Solution.** Write $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination,

$$
a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (1)
$$

It gives the linear system

$$
a = 1, \quad a - b = 0, \quad 2a + 3b + c = 0.
$$

Hence $a = b = 1$ and $c = -5$. **Note.** We can also solve $a, b, c$ by row reducing the augmented matrix of $[1]$, $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix}$.

By the properties of linear transformations,

$$
T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = T \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = T \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) - 5T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -10 \end{bmatrix}.
$$

**Note.** Since $T(\vec{e}_2) = T(- \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, we have $T(\vec{x}) = A\vec{x}$ with $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 6 & 2 \end{bmatrix}$. We can also solve $A = (a_{ij})_{i=1,2,3; j=1,2,3}$ by solving the system of 6 linear equations of $a_{ij}$ given by the problem. We can also solve $A = [\vec{a}_1 \vec{a}_2 \vec{a}_3]$ by solving the system

$$
\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad 0\vec{a}_1 - \vec{a}_2 + 3\vec{a}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 0\vec{a}_1 + 0\vec{a}_2 + \vec{a}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
$$
5. Circle the correct answers. No justifications necessary.

(a) True or [false]? A $5 \times 7$ matrix has seven rows.

   **Solution.** It has 5 rows

(b) [True] or false? The matrix

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

is in reduced echelon form.

(c) True or [false]? In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

   **Solution.** The reduced echelon form is unique

(d) True or [false]? A linear system is unique if and only if it has exactly one solution.

   **Solution.** At most one solution.

   **Note.** The textbook is vague in the definition of uniqueness. On page 7 it says that, “if a solution exists, it is the only one.” However, existence is not a prerequisite for uniqueness. **Uniqueness is the same as one-to-one.** In research, we often prove uniqueness without knowing existence.

(e) True or [false]? The equation $A\vec{x} = \vec{b}$ is consistent if the augmented matrix

\[
\begin{bmatrix}
A & \vec{b}
\end{bmatrix}
\]

has a pivot position in every row.

   **Solution.** The pivot position can not be in the augmentation column.
(f) True or false? A homogeneous linear system is always consistent.

(g) True or false? The columns of any $4 \times 5$ matrix are linearly dependent.

(h) True or false? Let $A$ be an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ be a nonzero vector. If the solution set of $A\vec{x} = \vec{0}$ is infinite, then the solution set of $A\vec{x} = \vec{b}$ is also infinite.

Solution. It could be empty.

(i) True or false? The transform $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ y - x \\ 3x \end{bmatrix}$ is linear.

(j) True or false? If $A$ is an $m \times n$ matrix, then the range of the transformation $x \mapsto Ax$ is $\mathbb{R}^m$.

Solution. The range could be a strict subset of $\mathbb{R}^m$. 