The University of British Columbia
Final Examination - April 20, 2007

Mathematics 221
Sections 201, 202, 203
Instructors: Dr. Macasieb, Dr. Tsai, and Dr. Liu

Closed book examination  Time: 2.5 hours

Name ___________________________  Signature ___________________________

Student Number __________________

Special Instructions:
- Be sure that this examination has 12 pages. Write your name on top of each page.
- No calculators or notes are permitted.
- Show all your work. Unsupported solutions deserve no mark.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

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1. [12pt] Consider the following linear system

\[
\begin{align*}
    x + 3y - 2z + 2w &= 1 \\
    y + z - 2w &= 2 \\
    x + 2y - 2z + aw &= 0 \\
    2x + 8y - z + w &= b
\end{align*}
\]

For which values of \( a \) and \( b \), if any, does the system have: (Justify your answers!!)

(i) No solution?  
(ii) Exactly one solution?  
(iii) Exactly two solutions?  
(iv) More than two solutions?
2. [10pt] Let $S$ be the map in $\mathbb{R}^3$ which rotates points about the $x_1$-axis by an angle $\pi/2$ (the axes are oriented by the right hand rule). Let $T$ be the map in $\mathbb{R}^3$ which translates points by the formula $T(x_1, x_2, x_3)^T = (x_1 + 1, x_2 - 1, x_3)^T$. One of them is a linear transformation and the other is not.

(i) Decide and justify which one is NOT a linear transformation.
(ii) You may assume the other one is a linear transformation. Find its standard matrix.
3. [10pt] For what values of $k$ is the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & k \end{bmatrix}$ invertible? When it is invertible, find its inverse.
4. [12pt] Let \( W = \left\{ \begin{bmatrix} b + 2c - d \\ 2b + 4c - d \\ d \\ -b - 2c + d \end{bmatrix} \mid b, c, d \text{ real} \right\} \).

(i) Find a matrix \( A \) such that \( \text{Col } A = W \).

(ii) Find a basis for \( W \).

(iii) If \( B = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & k \\ 1 & 1 & 1 & 3 \end{bmatrix} \) and \( \dim(\text{Row } B) = 2 \), find the value of the constant \( k \).
5. [10pt] Let \( A = \begin{bmatrix} x & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 1 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & x \end{bmatrix} \). Find all values of \( x \) such that \( A \) is not invertible.
6. [12pt] Let \( \mathbb{P}_2 \) be the vector space of polynomials of degree at most 2.

(i) The set \( B = \{1 + t, 1 + t^2, t + t^2\} \) is a basis for \( \mathbb{P}_2 \). Find the coordinate vector \([2 + t - t^2]_B\).

(ii) The set \( C = \{1 + t^2, t + t^2, 1 + t\} \) is also a basis for \( \mathbb{P}_2 \). Find \( \vec{p}(t) \) in \( \mathbb{P}_2 \) such that \( \vec{p}(1) = 1 \) and \([\vec{p}(t)]_B = [\vec{p}(t)]_C\).

(This problem is not covered.)
7. [7pt] Suppose a $2 \times 2$ matrix $A$ has eigenvalues 1 and $1/2$ with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$ 

What is $\lim_{k \to \infty} A^k$?
8. [12pt] Suppose
\[ \vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \]

Let \( W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} \).
(i) Determine the dimension of \( W \) and find a basis for \( W \).
(ii) Find an orthogonal basis for \( W \), and the orthogonal projection of \( \vec{y} \) onto \( W \).
(iii) What is the shortest distance from \( \vec{y} \) to \( W \)?
9. [8/2/5pt] The matrix \( M = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \).

(i) Verify that \( M \) has eigenvalues 0 and 3, and find the corresponding eigenspaces.
(ii) What is the rank of \( M \)?
(iii) Is \( M \) diagonalizable? Is there an orthogonal set of eigenvectors of \( M \) that forms a basis of \( \mathbb{R}^3 \)? Justify your answers.