The University of British Columbia
Final Examination - April 20, 2007

Mathematics 221
Sections 201, 202, 203
Instructors: Dr. Macasieb, Dr. Tsai, and Dr. Liu

Closed book examination  Time: 2.5 hours

Name ___________________________  Signature ___________________________

Student Number ________________

Special Instructions:
- Be sure that this examination has 12 pages. Write your name on top of each page.
- No calculators or notes are permitted.
- Show all your work. Unsupported solutions deserve no mark.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
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1. [12pt] Consider the following linear system

\[
\begin{align*}
    x + 3y - 2z + 2w &= 1 \\
    y + z - 2w &= 2 \\
    x + 2y - 2z + aw &= 0 \\
    2x + 8y - z + w &= b
\end{align*}
\]

For which values of \(a\) and \(b\), if any, does the system have: (Justify your answers!!)

(i) No solution?  
(ii) Exactly one solution?  
(iii) Exactly two solutions?  
(iv) More than two solutions?

**Solution**  

The augmented matrix

\[
\begin{bmatrix}
    1 & 3 & -2 & 2 & 1 \\
    0 & 1 & 1 & -2 & 2 \\
    1 & 2 & -2 & a & 0 \\
    2 & 8 & -1 & 1 & b
\end{bmatrix} \sim \begin{bmatrix}
    1 & 3 & -2 & 2 & 1 \\
    0 & 1 & 1 & -2 & 2 \\
    0 & -1 & 0 & a - 2 & -1 \\
    0 & 2 & 3 & -3 & b - 2
\end{bmatrix} \sim \begin{bmatrix}
    1 & 3 & -2 & 2 & 1 \\
    0 & 1 & 1 & -2 & 2 \\
    0 & 0 & 1 & a - 4 & 1 \\
    0 & 0 & 1 & b - 6 &
\end{bmatrix} \sim \begin{bmatrix}
    1 & 3 & -2 & 2 & 1 \\
    0 & 1 & 1 & -2 & 2 \\
    0 & 0 & 0 & 5 - a & b - 7
\end{bmatrix}
\]

Since the first three rows have 3 pivot positions, the number of solutions is decided by the last row:

(i) The last row corresponds to \((5 - a)w = b - 7\). There is no solution if \(a = 5\) but \(b \neq 7\).
(ii) There is exactly one solution if there are 4 pivot positions, i.e., when \(a \neq 5\).
(iii) It never happens for a linear system to have exactly two solutions.
(iv) There are multiple solutions if the last equation is void, i.e., when \(a = 5\) and \(b = 7\).
2. [10pt] Let $S$ be the map in $\mathbb{R}^3$ which rotates points about the $x_1$-axis by an angle $\pi/2$ (the axes are oriented by the right hand rule). Let $T$ be the map in $\mathbb{R}^3$ which translates points by the formula $T(x_1, x_2, x_3)^T = (x_1 + 1, x_2 - 1, x_3)^T$. One of them is a linear transformation and the other is not.

(i) Decide and justify which one is NOT a linear transformation.

(ii) You may assume the other one is a linear transformation. Find its standard matrix.

**Solution**

(i) The translation $T$ is not a linear transformation since

$$T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

(ii) Suppose the rotation $S$ is a linear transformation. The column vectors of its standard matrix is given by the images of the standard basis. Since

$$S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad S \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad S \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$

the standard matrix of $S$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$
3. [10pt] For what values of \( k \) is the matrix \( A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & k \end{bmatrix} \) invertible? When it is invertible, find its inverse.

Solution \[
\begin{align*}
\begin{bmatrix} A \mid I \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & k & 0 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & k & -1 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/k & 1/k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{align*}
\]

Thus \( A \) is invertible iff \( k \neq 0 \). In this case,

\[
\begin{align*}
\begin{bmatrix} A \mid I \end{bmatrix} &\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/k & 1/k \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/k & 1/k \end{bmatrix} = [I \mid A^{-1}]
\end{align*}
\]

that is, when \( k \neq 0 \),

\[
A^{-1} = \begin{bmatrix} 1/k & 1 & -1/k \\ 1 & 0 & 0 \\ -1/k & 0 & 1/k \end{bmatrix}
\]
4. [12pt] Let $W = \left\{ \begin{bmatrix} b + 2c - d \\ 2b + 4c - d \\ -b - 2c + d \end{bmatrix} \bigg| b, c, d \text{ real} \right\}$. 

(i) Find a matrix $A$ such that $\text{Col } A = W$.

(ii) Find a basis for $W$.

(iii) If $B = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & k \\ 1 & 1 & 1 & 3 \end{bmatrix}$ and $\dim (\text{Row } B) = 2$, find the value of the constant $k$.

**Solution**

(i) 

\[
\begin{bmatrix} b + 2c - d \\ 2b + 4c - d \\ -b - 2c + d \end{bmatrix} = b \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

Hence, if $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 4 & -1 \\ 0 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix}$, then $\text{Col } A = W$.

(ii) 

\[
A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ 0 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Thus, \( \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \) is a basis for $W$.

(iii) Reduce the matrix $B$:

\[
B = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & k \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & k \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & k - 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

In order to have $\dim (\text{Row } B) = 2$, $k - 2 = 0$ or $k = 2$. 

5. [10pt] Let \( A = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix} \). Find all values of \( x \) such that \( A \) is not invertible.

**Solution**

\[
\det A = \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = \begin{vmatrix} x+4 & x+4 & x+4 & x+4 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \end{vmatrix} = (x+4)(x-1)^4.
\]

In the second equality we replaced the first row by the sum of all rows.
In the fourth equality we subtracted the first row from all other rows.
The last equality is because the matrix is upper triangular.
Hence \( A \) is not invertible iff \( \det A = 0 \) iff \( x = -4 \) or \( 1 \).
6. [12pt] Let \( P_2 \) be the vector space of polynomials of degree at most 2.

(i) The set \( B = \{1 + t, 1 + t^2, t + t^2\} \) is a basis for \( P_2 \). Find the coordinate vector \([2 + t - t^2]_B\).

(ii) The set \( C = \{1 + t^2, t + t^2, 1 + t\} \) is also a basis for \( P_2 \). Find \( \vec{p}(t) \) in \( P_2 \) such that \( \vec{p}(1) = 1 \) and \( [\vec{p}(t)]_B = [\vec{p}(t)]_C \).

(This problem is not covered.)

**Solution**

(i) Let \( [2 + t - t^2]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \). Then

\[
2 + t - t^2 = c_1(1 + t) + c_2(1 + t^2) + c_3(t + t^2) = (c_1 + c_2) + (c_1 + c_3)t + (c_2 + c_3)t^2
\]

Thus \( \begin{cases} c_1 + c_2 = 2 \\ c_1 + c_3 = 1 \\ c_2 + c_3 = -1 \end{cases} \), which implies that \( \begin{cases} c_1 = 2 \\ c_2 = 0 \\ c_3 = -1 \end{cases} \).

Hence \( [2 + t - t^2]_B = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \).

(ii) Let \( [\vec{p}(t)]_B = [\vec{p}(t)]_C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \). Then

\[
c_1(1 + t) + c_2(1 + t^2) + c_3(t + t^2) = \vec{p}(t) = c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + t),
\]

which implies that \( c_1 = c_2 = c_3 \). Hence we get

\[
\vec{p}(t) = c_1(1 + t) + c_1(1 + t^2) + c_1(t + t^2) = c_1(2 + 2t + 2t^2).
\]

Using \( \vec{p}(1) = 1 \), we get \( 1 = \vec{p}(1) = c_1(2 + 2 + 2) \), or \( c_1 = \frac{1}{6} \). Thus \( \vec{p}(t) = \frac{1}{3} + \frac{1}{3}t + \frac{1}{3}t^2 \).
7. [7pt] Suppose a $2 \times 2$ matrix $A$ has eigenvalues 1 and $1/2$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

What is $\lim_{k \to \infty} A^k$?

**Solution**  Since $A$ has 2 distinct eigenvalues, $A$ is diagonalizable, i.e., $A = PDP^{-1}$, where $P = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$. Therefore, $\lim_{k \to \infty} A^k = \lim_{k \to \infty}(PDP^{-1})^k = \lim_{k \to \infty} PD^kP^{-1}$. Since $D^k = \begin{bmatrix} 1^k & 0 \\ 0 & (1/2)^k \end{bmatrix}$, we have $\lim_{k \to \infty} D^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore

$$\lim_{k \to \infty} A^k = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 15 & -5 \end{bmatrix}.$$
8. [12pt] Suppose
\[ \vec{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} -1 \\ 1 \\ -7 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}. \]

Let \( W = \text{Span}\{ \vec{w}_1, \vec{w}_2, \vec{w}_3 \} \).

(i) Determine the dimension of \( W \) and find a basis for \( W \).
(ii) Find an orthogonal basis for \( W \), and the orthogonal projection of \( \vec{y} \) onto \( W \).
(iii) What is the shortest distance from \( \vec{y} \) to \( W \)?

**Solution**

(i) Form the matrix
\[ A = [ \vec{w}_1 \ \vec{w}_2 \ \vec{w}_3 ] = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 7 \end{bmatrix}, \]
and we row reduce to determine if the columns of \( A \) are linearly dependent. Since
\[ \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ \vec{w}_3 = 2\vec{w}_1 - 3\vec{w}_2. \] So \( W = \text{Span}\{ \vec{w}_1, \vec{w}_2 \} \). Since \( \vec{w}_1 \) and \( \vec{w}_2 \) are not multiples, they are linearly independent. This shows that \( \{ \vec{w}_1, \vec{w}_2 \} \) is a basis for \( W \) and \( \dim W = 2 \).

(ii) Since \( \vec{w}_1 \cdot \vec{w}_2 = 0 \), \( \{ \vec{w}_1, \vec{w}_2 \} \) is an orthogonal basis. Using this orthogonal basis, the orthogonal projection of \( \vec{y} \) onto \( W \) is
\[ \text{proj}_W \vec{y} = \frac{\vec{w}_1 \cdot \vec{y}}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{y}}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2 = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}. \]

(iii) The shortest distance from \( \vec{y} \) to \( W \) is \( ||\vec{y} - \text{proj}_W \vec{y}|| \). Since
\[ \vec{y} - \text{proj}_W \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}, \]
the distance is
\[ ||\vec{y} - \text{proj}_W \vec{y}|| = \frac{\sqrt{2}}{2}. \]
9. [8/2/5pt] The matrix \( M = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \).

(i) Verify that \( M \) has eigenvalues 0 and 3, and find the corresponding eigenspaces.

(ii) What is the rank of \( M \)?

(iii) Is \( M \) diagonalizable? Is there an orthogonal set of eigenvectors of \( M \) that forms a basis of \( \mathbb{R}^3 \)? Justify your answers.

**Solution**

(i) 
\[
M - 3I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

Therefore, there exist nonzero elements in \( \text{Nul} \ M - 3I \) and \( \lambda = 3 \) is an eigenvalue of \( M \). The set of eigenvectors corresponding to \( \lambda = 3 \) are the set of vectors in \( \text{Nul} \ M - 3I \):

\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},
\]

\( x_2, x_3 \) are free. Therefore \( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \) are a basis of eigenvectors corresponding to \( \lambda = 3 \).

\( M = M - 0I = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \). So \( M\vec{x} = \vec{0} \) has nontrivial solutions, and \( \lambda = 0 \) is an eigenvalue of \( M \). \( \text{Nul} \ M \) is the corresponding eigenspace. Now, \( \text{Nul} \ M \) is the set of vectors

\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
\]

where \( x_3 \) is free. So \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector corresponding to \( \lambda = 0 \).

(ii) By part (i), \( \dim \text{Nul} \ M = 1 \), and the Rank Theorem states that for an \( m \times n \) matrix \( M \), \( \text{rank} \ M + \dim \text{Nul} \ M = n \). So, \( \text{rank} \ M = 3 - 1 = 2 \).

(iii) Since \( M \) has a basis of eigenvectors of \( \mathbb{R}^3 \), \( M \) is diagonalizable. Furthermore, since \( M \) is symmetric, there is an orthogonal set of eigenvectors that forms a basis of \( \mathbb{R}^3 \).

Note: A square matrix \( M \) is symmetric if \( M^T = M \). The last sentence is a theorem in the chapter of “Symmetric matrices and quadratic forms” which we did not cover this term.

Without using it, for the matrix \( M \) we can construct an orthogonal basis explicitly by:

\[
\vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \text{proj}_{\vec{u}_1} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

Note \( \vec{u}_2 \) is still an eigenvector. We can also normalize them to get an orthonormal basis.