1. Solve the following initial value problem (i.e. a differential equation with an initial condition.)

\[
\begin{align*}
    y' &= \sec(y), \\
    y(0) &= 0.
\end{align*}
\]

*Hint*: \(\sec(y) = 1/\cos(y)\).

*Solution*. We have

\[
    \frac{dy}{dx} = \sec(y) = \frac{1}{\cos(y)}.
\]

By separation of variables, we have

\[
    \cos(y)dy = dx.
\]

Integrating on both sides,

\[
    \sin(y) = x + C,
\]

so \(y = \arcsin(x + C)\). Using the initial condition \(y(0) = 0\), we have \(C = 0\), so

\[
    y = \arcsin(x).
\]

2. Find the infinite sum

\[
    S = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \frac{16}{243} + \cdots.
\]

*Solution*. By observation, we notice that the \(n\)-th summand can be written as

\[
    a_n = \frac{2^{n-1}}{3^n}.
\]

Thus \(S\) is a geometric series with common ratio \(r = a_2/a_1 = 2/3\) which has \(|r| < 1\), with the initial term \(a_1 = 1/3\).

Hence the formula for geometric series gives

\[
    S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1 - r} = \frac{1/3}{1 - 2/3} = 1.
\]
3. Consider a continuous random variable $X$ with the probability density function

$$f(x) = \begin{cases} 
\frac{c}{1 + x^2}, & \text{if } x \geq 0, \\
0, & \text{if } x < 0.
\end{cases}$$

(a) Find $c$.

(b) Find the cumulative distribution function $F$.

(c) Does $E(X)$ exist as a real number? If yes, find $E(X)$. If not, state why.

**Solution.** (a) For $f$ to be a probability density function, we must have

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

In this case,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \frac{c}{1 + x^2}dx = \lim_{b \to \infty} \int_{b}^{\infty} \frac{c}{1 + x^2}dx = \lim_{b \to \infty} \arctan(x) \bigg|_{x=b}^{x=0} = \lim_{b \to \infty} \arctan(b) = \frac{\pi}{2}.$$

Since $\int_{-\infty}^{\infty} f(x)dx = 1$, we have $c = 2/\pi$.

(b) By definition,

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Hence if $x < 0$, then $F(x) = 0$. If $x \geq 0$, then

$$F(x) = \int_{0}^{x} \frac{2}{\pi} \frac{1}{1 + t^2}dt = \frac{2}{\pi} \arctan(t) \bigg|_{t=0}^{t=x} = \frac{2}{\pi} \arctan(x).$$

As a result,

$$F(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{2}{\pi} \arctan(x), & \text{if } x \geq 0.
\end{cases}$$

(c) By definition,

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{\infty} \frac{2}{\pi} \frac{x}{1 + x^2}dx.$$

Changing $u = 1 + x^2$, we have $du = 2xdx$, and thus

$$\int \frac{2}{\pi} \frac{x}{1 + x^2}dx = \int \frac{1}{\pi} \frac{1}{u}du = \frac{1}{\pi} \ln |u| + C = \frac{1}{\pi} \ln |1 + x^2| + C = \frac{1}{\pi} \ln(1 + x^2) + C.$$
Hence

\[ \int_0^\infty \frac{2}{\pi} \frac{x}{1 + x^2} \, dx = \lim_{b \to \infty} \int_0^b \frac{2}{\pi} \frac{x}{1 + x^2} \, dx \]

\[ = \lim_{b \to \infty} \frac{1}{\pi} \ln(1 + b^2) \bigg|_{x=0}^{x=b} \]

\[ = \lim_{b \to \infty} \frac{1}{\pi} \ln(1 + b^2) \]

\[ = \infty. \]

Hence the integral defining \( E(X) \) does not converge, so \( E(X) \) does not exist as a real number.

\( \square \)