More Ratio Test Examples

\[
\sum_{n=1}^{\infty} \frac{n!}{n^{2n}}
\]

Sol:

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{2(n+1)}} \div \frac{n!}{n^{2n}}
\]

\[
= \frac{(n+1)!}{(n+1)^{2(n+1)}} \times \frac{n^{2n}}{n!}
\]

\[
= \frac{(n+1)!}{(n+1)^{2n+2}} \times \frac{n^{2n}}{n!}
\]

\[
= \frac{n^{2n}}{(n+1)^{2n+2}} \times \frac{1}{n+1}
\]

\[
= \frac{1}{(\frac{n+1}{n})^{2n}} \times \frac{1}{n+1}
\]

\[
= \left[ \frac{1}{(1+\frac{1}{n})^n} \right]^2 \times \frac{1}{n+1}
\]
As $n \to \infty$, using $(1 + \frac{1}{n})^n \to e$,

$$\lim_{n \to \infty} \left[ \frac{1}{(1 + \frac{1}{n})^n} \right]^2 \cdot \frac{1}{n+1} = \left( \frac{1}{e} \right)^2 \cdot 0 = 0$$

Since $0 < 1$, the series $\sum_{n=1}^{\infty} \frac{n!}{n^{2n}}$ converges.

Two Examples when ratio test is inconclusive.

**e.g. 1:** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test, as $p=2>1$.

However, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{(n+1)^2} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1$.

**e.g. 2:** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the integral test, as $p=1 \leq 1$.

However, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{n+1} = \lim_{n \to \infty} \frac{n}{n+1} = 1$.

In both cases, we have $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$, but one converges and the other diverges.
# A Short Review

## Tests

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<tbody>
<tr>
<td>Divergence Test</td>
<td>None</td>
<td>- If $\sum a_n$ conv., then $a_n \to 0$; equivalently, if $a_n \not\to 0$, then $\sum a_n$ diverges.</td>
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| Comparison Test    | $0 < a_n \leq b_n$ | - If $\sum b_n < 0$, then $\sum a_n < 0$. | - If $\sum a_n = 0$, then $\sum b_n = 0$. |

| Limit Comparison Test | $a_n, b_n > 0$, $\lim_{n \to \infty} \frac{a_n}{b_n} = L$ | - If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge, or both diverge. | - If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges. | - If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. |

| Ratio Test   | $a_n > 0$, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$ | - If $0 < L < 1$, then $\sum a_n$ conv. | - If $L > 1$, then $\sum a_n$ div. | - If $L = 1$, no conclusion. |

| Integral Test | $a_n \geq 0$, $f(n) = a_n$ and $f(x) \downarrow 0$ as $x \to \infty$. (After some finite $n_0$) | $\sum a_n$ converges if and only if $\int_0^{\infty} f(x) \, dx < \infty$ for some $n_0$ large. |