Solution:

1. By definition of p.d.f., we have
\[ \int_{-\infty}^{\infty} f(x) \, dx = 1. \]
Hence
\[ c \int_{0}^{\infty} e^{-2x} \, dx = 1, \]
or
\[ c = \frac{1}{\int_{0}^{\infty} e^{-2x} \, dx} = \frac{1}{-\frac{1}{2} e^{-2x}\bigg|_{0}^{\infty}} = \frac{1}{-\frac{1}{2} \left(\lim_{b \to \infty} e^{-2b} - 1\right)} = \frac{1}{-\frac{1}{2} (0-1)} = 2. \]

2. By definition of c.d.f.,
\[ F(x) = \int_{-\infty}^{x} f(t) \, dt \]

\[ = \int_{0}^{x} 2 e^{-2t} \, dt = -e^{-2t}\bigg|_{t=0}^{t=x} = 1 - e^{-2x}, \]
if \( x > 0 \)
\[ = 0, \]
if \( x = 0 \).
3) Mean = \( \int_{-\infty}^{\infty} x f(x) \, dx \)

= \( \int_{0}^{\infty} x \cdot 2e^{-2x} \, dx \)

Integration by parts:

\( u = 2x, \quad v' = e^{-2x} \)

\( u' = 2, \quad v = -\frac{1}{2} e^{-2x} \)

The above:

\( 2x \left( -\frac{1}{2} e^{-2x} \right) - \int 2 \cdot \left( -\frac{1}{2} \right) e^{-2x} \, dx \)

Hence:

\( \int 2x e^{-2x} \, dx = -xe^{-2x} + \int e^{-2x} \, dx \)

= \( -xe^{-2x} - \frac{1}{2} e^{-2x} + C \).

Hence:

\( \int_{0}^{\infty} 2x e^{-2x} \, dx = \lim_{b \to \infty} \left( -be^{-2b} - \frac{1}{2} e^{-2b} \right) - \left( -\frac{1}{2} e^{-2x} \right) \bigg|_{x=0}^{\frac{1}{2}} \)

= \( -\lim_{b \to \infty} be^{-2b} + \frac{1}{2} \).

\( \lim_{b \to \infty} be^{-2b} = \lim_{b \to \infty} \frac{b}{e^{2b}} \quad \text{L'Hôpital's Rule} \quad \lim_{b \to \infty} \frac{1}{2be^{2b}} = 0 \)

Hence:

\( \int_{0}^{\infty} = \frac{1}{2} \).
\[ \text{Var}(X) = E(X^2) - (E(X))^2. \]

Now \( (E(X))^2 = \left( \frac{1}{2} \right)^2 = \frac{1}{4} \) by 3.

Need to find \( E(X^2) \):

\[
E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx
\]

\[
= \int_{0}^{\infty} x^2 \cdot 2e^{-2x} \, dx
\]

Integration by parts again:

\[
u = 2x^2 \quad v' = e^{-2x}
\]

\[
u' = 4x \quad v = -\frac{1}{2} e^{-2x}
\]

\[
\int 2x^2 e^{-2x} \, dx = \left[ (2x^2) \left( -\frac{1}{2} e^{-2x} \right) \right] - \int 4x \left( -\frac{1}{2} e^{-2x} \right) \, dx
\]

\[
= -x^2 e^{-2x} + \left[ \int 2x e^{-2x} \, dx \right]
\]

\[
= -x^2 e^{-2x} - xe^{-2x} - \frac{1}{2} e^{-2x} + C.
\]

Hence

\[
\int_{0}^{\infty} 2x^2 e^{-2x} \, dx = \left( \lim_{b \to 0^+} -b^2 e^{-2b} - be^{-2b} - \frac{1}{2} e^{-2b} \right)
\]

\[
- \left( -x^2 e^{-2x} - xe^{-2x} - \frac{1}{2} e^{-2x} \right) \bigg|_{x = 0}
\]

\[
= \left( -\lim_{b \to 0^+} b^2 e^{-2b} + be^{-2b} \right) + \frac{1}{2}
\]

\[
= \frac{1}{2}
\]

(Using L'Hôpital's Rule again)
Therefore

\[
\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.
\]

\[
\sigma(X) = \sqrt{\text{Var}(X)} = \sqrt{\frac{1}{4}} = \frac{1}{2}.
\]

\[5\]

\[
\Pr(1 \leq X \leq 2) = \int_1^2 f(x) \, dx
\]

\[
= \int_1^2 2e^{-2x} \, dx
\]

\[
= -e^{-2x}\bigg|_1^2 = -e^{-4} - (-e^{-2})
\]

\[
= e^{-2} - e^{-4}. (\approx 11.7\%)
\]