Announcements

1. Midterm II: Mean ≈ 34/50
2. Quiz III: Next Friday (29th) (Mr. Man Chan Lee will come and substitute me for the half lecture)
3. Webwork 9 is due on Mar 21.

Today: Some new concepts of probability
- Introduction to our last chapter: Sequences and Series.
Last time: c.d.f. \( F(x) \)
p.d.f. \( f(x) \)

\[ F'(x) = f(x) \quad \text{and} \quad F(x) = \int_{-\infty}^{x} f(t) \, dt \]

Today: \begin{itemize}
  \item Mean (expectation)
  \item Variance and standard deviation
\end{itemize}

**Definition 1.** Let \( X \) be a random variable and suppose it has p.d.f. \( f \).
Then we defined the mean (expectation) of \( X \) by

\[ E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx. \]

\( \Rightarrow \) the likelihood of \( X \) taking values around \( x \).
e.g. Let $X$ be the random variable defined by choosing a real number uniformly randomly between $-1$ and $1$.

Then the p.d.f. is given by:

$$f(x) = \begin{cases} 
\frac{1}{2}, & \text{if } |x| \leq 1 \\
0, & \text{if } |x| > 1
\end{cases}$$

By def. $E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$

$$= \int_{-1}^{1} x \cdot \frac{1}{2} \, dx$$

$$= \frac{1}{4} x^2 \bigg|_{x=-1}^{x=1} = 0$$
Variance and Standard Deviation.

Definition 2. Let $X$ be a random variable and $f$ be its p.d.f. Let $E(X)$ be its expectation.

We define the variance of $X$ by

$$\text{Var}(X) = E\left((X - E(X))^2\right).$$

$$= \int_{-\infty}^{\infty} (x - E(X))^2 f(x) \, dx$$

Prop. Equivalently,

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x) \, dx - (E(X))^2.$$
The standard deviation of $X$, denoted $\sigma(X)$, is defined by

$$\sigma(X) = \sqrt{\text{var}(X)}.$$ 

E.g. 2. Using e.g. 1. $f(x) = \begin{cases} \frac{1}{2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

$E(X) = 0$. Find $\text{Var}(X)$, $\sigma(X)$.

Sol: To find $\text{Var}(X)$, we use

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= E(X^2) - 0, \text{ since } E(X) = 0$$

$$= \int_{-\infty}^{\infty} x^2 f(x) \, dx$$

$$= \int_{-1}^{1} x^2 \cdot \frac{1}{2} \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{3} x^3 \bigg|_{-1}^{1} = \frac{1}{3}.$$
\[ \sigma(X) = \sqrt{\text{Var}(X)} = \frac{1}{3} = \frac{1}{\lambda}. \]

Exercise: Let \( X \) be a random variable with p.d.f \( f(x) \): (\text{Exponential random variable}) \( f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \) for some constant \( \lambda \).

1. Find the value of \( \lambda \).
2. Find the c.d.f.
3. Find the mean and
4. Find the variance and standard deviation.
5. Find the prob. \( \Pr(1 \leq X \leq 2) \).
Sequences and Series.

Definition: A sequence is an infinite ordered list of numbers. It continues forever.

Infinite: e.g. 1, 2, 3, 4, 5, 6, ... is a sequence.

Ordered: e.g. 1, 2, 3, 4, 5, 6, 7, 8 is not a sequence.

Ways to define sequences:
A. By a formula; B. A description.
C. A List (enumeration)
D. A recursive definition (iterative)

e.g. \( a_n = \frac{\sin(\ln(n))}{\exp(\arctan(\frac{n^2}{n}))} \)

The \( n \)-th term in a sequence \( \{a_n\} \).

\( a_n = n^2 \). This means the sequence \( \uparrow: 1, 4, 9, 16, 25, \ldots \)

"The \( n \)-th term of the sequence \( a_n \) is defined by \( n^2 \)."

Convention: In this course, \( n \) always starts with 1. (not 0)