Midterm Covers up to today's lecture (Numerical integration and error analysis)

Objectives for today:
- Error and error analysis
- Improper integrals ?? (Not in MT II)

Last time, we defined 3 kinds of numerical integration scenarios. We'll analyse their errors (more precisely, the maximum/the worst error)
Error:

Suppose you estimate some quantity whose exact value is \( E \). Let your approximating value by \( A \).

Then we define:

- The absolute error \( = |A - E| \)
- The relative error \( = \frac{|A - E|}{E} \)

E.g. Given that:
- The radius of a basketball is roughly 0.1 m.
- The radius of a ping-pong ball is roughly 0.01 m.
Q1: If we use the radius of a basketball to estimate the radius of a ping-pong ball, what are the absolute error and relative error, respectively?

A: Absolute error

\[ A = |A - E| \]

\[ = |0.1 - 0.09| \]

\[ = 0.01 \text{ m} \]

Relative error

\[ \text{Relative error} = \frac{|A - E|}{E} = \frac{0.01}{0.01} = 9 = 900\% \]
Q2: If we use the radius of a ping-pong ball to estimate the radius of a basketball, what are the absolute error and relative error, respectively?

Ans: Absolute error

\[ |A - E| = |0.01 - 0.11| = 0.09 \]

Relative error

\[ \frac{|A - E|}{E} = \frac{0.09}{0.1} = 0.9 = 90\% \]
Theorem (Error Bounds)

1. Suppose \( f \) is a function defined on \([a, b]\) such that \( |f''(x)| \leq K \)
   for all \( x \) between \( a \) and \( b \).

Then: if we use the Midpoint rule or the Trapezoidal rule to estimate \( \int_{a}^{b} f(x) \, dx \),
using \( n \) subintervals, the error bounds are given by:

\[
E_m \leq \frac{K(b-a)}{24} \cdot (\Delta x)^2 = \frac{K(b-a)}{24n^2},
\]

\[
E_T \leq \frac{K(b-a)}{12} \cdot (\Delta x)^2 = \frac{K(b-a)^3}{12n^2},
\]

respectively.

(By error bounds, I mean the error is GUARANTEED not to exceed the bound)
2. Suppose \( f \) is function defined on \([a,b]\) such that \( |f^{(4)}(x)| \leq K \) for all \( a \leq x \leq b \).

Then: if we use the Simpson's rule to estimate \( \int_a^b f(x) \, dx \), using \( n \) subintervals, the error bound is given by

\[
E_s \leq \frac{K (b-a)}{180} (\Delta x)^4
\]

\[
= \frac{K (b-a)^5}{180 n^4}.
\]

(Better than \( E_M \), \( E_T \) as \( n \) gets large)
Eg.1.
Suppose we will estimate
\[ \int_{3}^{4} \frac{1}{x} \, dx \]
by Simpson's rule.
If we take \( n = 5 \), what is the worst error it could be?

Sol:
Step 1: Look up the error bound formula for Simpson's rule
( In the formula sheet )
\[ E_S \leq \frac{k(b-a)^5}{180n^4} \]
Step 2: Identify \( a, b, n \).
\( a = 3, \ b = 4, \ n = 5 \)
Step 3: Find \( K \).
One option is to compute the maximum of \( |f^{(4)}(x)| \) on \([a, b]\).
\[ f(x) = \frac{1}{x} = x^{-1}. \]
\[ f'(x) = -x^{-2}. \]
\[ f''(x) = 2x^{-3}. \]
\[ f'''(x) = -6x^{-4}. \]
\[ f^{(4)}(x) = 24x^{-5}. \]

Hence \[ |f^{(4)}(x)| = 24x^{-5}. \]

Mini-problem: Maximize \( 24x^{-5} \) on the interval \([a, b] = [3, 4] \).

We know if \( x = 3 \), \( \max |f^{(4)}(x)| = 24 \).

Hence take \( k = 24 \times 3^{-5} \times 3^{-5} \).

Step 4: Combine and computation:

\[
E_S \leq \frac{24 \times 3^{-5} \times (4-3)^5}{180 \times 5^4} \approx 5.85 \times 10^{-7}.
\]

(Ready for calculator.)
E.g. 2.
Suppose we want to estimate \( \int_{0}^{\frac{1}{2}} e^{x^2} \, dx \) using Trapezoidal rule.

Given the fact that \( |f''(x)| \leq 6 \)
for all \( 0 \leq x \leq \frac{1}{2} \) where \( f(x) = e^{x^2} \),
find the smallest number \( n \) such that the error is guaranteed not to exceed \( 10^{-4} \).

Sol: Step 1: Look up the error bound formula: \( E_T \leq \frac{k(b-a)^3}{12n^2} \).

Step 2: Identify \( a, b, n \).
\( a = 0, \ b = \frac{1}{2}, \ n \ (TBD) \)
Step 3: Find $K$. Using the fact given in the question, we can take $k = 6$. Otherwise, find $|f''(x)|$ and compute its maximum over $[0, \frac{1}{2}]$.

Step 4: Solve an inequality:

$$E_I \leq \frac{k(b-a)^3}{12n^2} \leq 10^{-4}$$

$$\Rightarrow 6 \cdot \left(\frac{1}{2}\right)^3 \leq 10^{-4}$$

$$\Rightarrow 6 \cdot \left(\frac{1}{2}\right)^3 \leq 12n^2 \cdot 10^{-4}$$

$$\Rightarrow n^2 \geq \frac{6 \cdot \left(\frac{1}{2}\right)^3}{12} \cdot 10^4 = \left(\frac{1}{2}\right)^4 \cdot 10^4$$

$$\Rightarrow n \geq \left(\frac{1}{2}\right)^2 \cdot 10^2 = 25.$$ 

Hence the smallest number of subintervals $= 25$. 
Improper Integrals

Def: An integral is called improper if either

1. The range of integration is unbounded
2. The integrand \( = f(x) \) is unbounded

\[ 1 \text{ e.g. } \int_{-\infty}^{0} e^{-x} \, dx. \]

\[ \int_{0}^{1} \frac{1}{x} \, dx. \]

Next lecture: how do we define these concepts,