Today:

1. Examples of finding absolute max/min on a closed and bounded region.

2. Introduction to the Lagrange multiplier method.

Example 2. \( f(x,y) = x^2 + y^2 \). Region \( R \) is the rectangle \(-1 \leq x \leq 1, \ -1 \leq y \leq 2\).

Sol: Step 1: Find critical points of \( f \).

\[ f_x = 2x, \quad f_y = 2y. \]

Setting \( f_x = 0 = f_y \), we get \( x = y = 0 \).

Hence, \((0,0)\) is a critical point.

and \( f(0,0) = 0^2 + 0^2 = 0 \).
Step 2: The boundary of $R$ is divided into 4 parts, $C_1, C_2, C_3, C_4$.

On $C_1$: we consider

$$f_1(y) = f(1, y) = 1^2 + y^2 = 1 + y^2.$$  
So we need to optimize $f_1(y)$ on the interval $-1 \leq y \leq 2$.

$$f_1(y)$$

$$\begin{array}{c}
\text{maximum of } f_1(y) \\
\text{attained at } y = 2 \\
f_1(2) = 4 + 1 = 5.
\end{array}$$

$$\begin{array}{c}
\text{minimum of } f_1(y) \\
\text{attained at } y = 0 \\
f_1(0) = 1 + 0^2 = 1.
\end{array}$$

On $C_2$: we consider

$$f_2(x) = f(x, 2) = x^2 + 2^2 = x^2 + 4.$$  

$$f_2(x)$$

$$\begin{array}{c}
\text{max of } f_2(x) \text{ at } x = \pm 1, \\
f_2(\pm 1) = 4 + 1 = 5.
\end{array}$$

$$\begin{array}{c}
\text{min of } f_2(x) \text{ at } x = 0 \\
f_2(0) = 4.
\end{array}$$
On C₃...
On C₄, ...

(Omitted, Exercise)
<table>
<thead>
<tr>
<th>(x,y)</th>
<th>(0,0)</th>
<th>(1,2)</th>
<th>(1,0)</th>
<th>(1,2)</th>
<th>(-1,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x,y)</td>
<td>0</td>
<td>5</td>
<td>1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>crit/bdy</td>
<td>crit.</td>
<td>bdy(C₁)</td>
<td>bdy(C₂)</td>
<td>bdy(C₂)</td>
<td>bdy(C₂)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(0,2)</th>
<th>(-1,0)</th>
<th>(-1,-1)</th>
<th>(0,-1)</th>
<th>(1,-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>bdy(C₂)</td>
<td>bdy(C₃)</td>
<td>bdy(C₃)</td>
<td>bdy(C₄)</td>
<td>bdy(C₄)</td>
</tr>
</tbody>
</table>

Comparing the values at all points,

Max of f(x,y) is attained at (1,2) and (-1,2), where f(x,y) = 5.

Min of f(x,y) is attained at (0,0), where f(x,y) = 0.
Example 3.

\[ f(x,y) = (x-1)^2 + (y-1)^2 \] on the triangle region.

Sol. Step 1: Find critical points.

\[ f_x = 2(x-1), \quad f_y = 2(y-1) \]

Setting \( f_x = 0 = f_y \), we get

\[ (x, y) = (1, 1) \]

so \((1, 1)\) is a critical point.

\[ f(1, 1) = 0. \]

Step 2: Consider each part of the boundary.

i.e. the triangle formed by \( C_1, C_2, C_3 \).

On \( C_1 \):

\[ f_1(y) = f(0, y) = (0-1)^2 + (y-1)^2 = 1 + (y-1)^2. \]

Hence \( f_1(y) \) attains a min. at \( y = 1 \),
\[ f_1(1) = 1; \quad f_1(y) \] attains a max. at \( y = 0 \) or \( 2 \),
\[ f_1(0) = f_1(2) = 2. \]

\[ \boxed{\text{Note: Please correct if mistaken.}} \]
On $C_2$.

\[ f_2(x) = (x-1)^2 + 1 \]

- $f_2$ attains a min. at $x = 1$.
- $f_2(1) = 1$
- $f_2$ attains a max at $x = 0$ or $x = 2$.
- $f_2(0) = f_2(2) = 2$.

On $C_3$.

\[ f_3(x) = f(x, 2-x) \]

\[ = (x-1)^2 + (2-x-1)^2 \]

\[ = (x-1)^2 + (1-x)^2 \]

\[ = 2(x-1)^2. \]

On $0 \leq x \leq 2$,

- $f_3$ attains a max at $x = 0$ or $x = 2$.
- $f_3(0) = 2 = f_3(2)$.
- $f_3$ attains a min at $x = 1$.
- $f_3(1) = 0$. 

(Using $x+y=2$)
<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>(1, 1)</th>
<th>(0, 1)</th>
<th>(0, 0)</th>
<th>(0, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x, y)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>critical/</td>
<td>crit.</td>
<td>bdy($C_1$)</td>
<td>bdy($C_1$)</td>
<td>bdy($C_1$)</td>
</tr>
<tr>
<td>boundary</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(1, 0)</th>
<th>(0, 0)</th>
<th>(2, 0)</th>
<th>(0, 2)</th>
<th>(2, 0)</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>bdy($C_2$)</td>
<td>bdy($C_2$)</td>
<td>bdy($C_2$), bdy($C_2$)</td>
<td>bdy($C_3$), bdy($C_2$)</td>
<td>bdy($C_3$), bdy($C_2$)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$f$ attains global max at 3 points:

(0, 0), (2, 0) and (0, 2), where $f(x, y) = 2$.

$f$ attains a global min. at (1, 1), where $f(x, y) = 0$. 
Example 4.

Optimize \( f(x, y) = x + y \) on the region \( R \) given by \( f(x, y): \ x^2 + y^2 \leq 4 \).

The disc of radius 2 centred at \((0, 0)\).

Sol: Step 1: \( f_x = 1, \ f_y = 1 \).
No critical points for \( f \) & \( b \).

Step 2: Optimize \( f(x, y) \) on the boundary \( x^2 + y^2 = 4 \) (*).

Attempt 1. Eliminate \( y \).

By (*), \( y = \pm \sqrt{4 - x^2} \), we have 2 pieces of the boundary.

For \( C_1 \),

\[ f_1(x) = f(x, \sqrt{4 - x^2}) = x + \sqrt{4 - x^2}, \ \text{for} \ -2 \leq x \leq 2 \]