Example of Implicit Differentiation, (multivariable case)

Example 1:

\[ \sin(xy) + e^z = z. \]

Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

Sol.: By implicit differentiation.

For \( x \),

\[ \frac{\partial}{\partial x} \sin(xy) + \frac{\partial}{\partial x} (e^z) = \frac{\partial}{\partial x} z. \]

Then,

\[ \begin{align*}
\text{Chain rule} & \quad \cos(xy) \cdot y + e^z \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \\
\text{algebra} & \quad \cos(xy) \cdot y = \frac{\partial z}{\partial x} (1 - e^z)
\end{align*} \]

\[ \downarrow \]

\[ \frac{\partial z}{\partial x} = \frac{\cos(xy) \cdot y}{1 - e^z}. \]
For $y$,
\[
\frac{\partial}{\partial y} \sin(xy) + \frac{\partial}{\partial y} (e^z) = \frac{\partial}{\partial y} z
\]
(Chain rule)
\[
\frac{\partial}{\partial y} \cos(xy) \cdot x + e^z \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}
\]
\[
\frac{\partial z}{\partial y} = \frac{\cos(xy) \cdot x}{1 - e^z}
\]

Example 2:
\[
x^2 + y^2 + xz + z^2 = 1 \quad (*)
\]
Find the point(s) on the surface above such that \( \frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y} \).

Sol: For $x$,
\[
\frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial x} (y^2) + \frac{\partial}{\partial x} (xz) + \frac{\partial}{\partial x} (z^2) = \frac{\partial}{\partial x} (1)
\]
(Prod rule)
\[
2x + 2y + 0 + 1 \cdot z + \frac{\partial z}{\partial x} \cdot x + 2z \cdot \frac{\partial z}{\partial x} = 0
\]
\( \frac{\partial z}{\partial x} (x + 2z) + (2x + z) = 0 \)

\[ \frac{\partial z}{\partial x} = -\frac{2x + z}{x + 2z} \]

For \( y \),

\[ \frac{\partial}{\partial y} (x^2) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial y} (xz) + \frac{\partial}{\partial y} (z^2) = \frac{\partial}{\partial y} \]

\[ 0 + 2y + x \cdot \frac{\partial z}{\partial y} + 2z \cdot \frac{\partial z}{\partial y} = 0 \]

\( (\text{algebra}) \)

\[ \frac{\partial z}{\partial y} (x + 2z) + 2y = 0 \]

\[ \frac{\partial z}{\partial y} = -\frac{2y}{x + 2z} \]
Solve: \[
\begin{cases}
\frac{2x + 2z}{x + 2z} = 0 \\
\frac{2y}{x + 2z} = 0
\end{cases}
\Rightarrow \begin{cases}
z = -2x \\
y = 0
\end{cases}
\]

Plugging in (*), we have.

\[x^2 + 0^2 + x(-2x) + (-2x)^2 = 1\]

\[\sqrt{3x^2} = 1\]

\[x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}}\]

There are 2 points on the surface (*) such that \[\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}\], namely,

\[P_1: \begin{cases} x = \frac{1}{\sqrt{3}} \\
y = 0 \\
z = -\frac{2}{\sqrt{3}} \end{cases} \quad P_2: \begin{cases} x = -\frac{1}{\sqrt{3}} \\
y = 0 \\
z = \frac{2}{\sqrt{3}} \end{cases}\]

Exercise: Classify \(P_1\) and \(P_2\) as critical points, a local max/min/saddle point.

Hint: For this, need to use 2nd order implicit differentiation.