Local Maximum: $f(x,y) \leq f(a,b)$

Special example:
A constant function has local maxima/minima at every point.

Local Minimum:

$f(x,y) \geq f(a,b)$

Local max

Local min
Def (Critical point)

A critical point \((a, b)\) for a function \(f\) is an interior point such that
\[
fx(a, b) = 0 = fy(a, b).
\]
Hence all local maxima/minima must be a critical point.

Example 1 (A primitive example)

\[
f(x, y) = x^2 + y^2.
\]

Sol: Step 1: Compute \(fx\) and \(fy\).

\[
fx(x, y) = 2x,
\]

\[
fy(x, y) = 2y
\]

Step 2: Using the necessary condition, we set
\[
\begin{cases}
fx(x, y) = 0 \\
fy(x, y) = 0
\end{cases}
\]

\[
\begin{cases}
2x = 0 \\
2y = 0
\end{cases} \rightarrow \begin{cases}
x = 0 \\
y = 0
\end{cases}
\]

The only critical point of \(f\) is \((0, 0, 0)\).

The local minimum of \(f\) is \((0, 0, 0)\).
Example 2: \( f(x,y) = x^2 - y^2 \).

Step 1: \( f_x = 2x, \quad f_y = -2y \)

Step 2: \( \begin{cases} f_x = 0 \Rightarrow x = 0 \\ f_y = 0 \Rightarrow y = 0 \end{cases} \)

So \((0,0)\) is a critical point of \( f \).

\((0,0,0)\)

But this point is neither a local max. nor local min.

Sufficient Conditions for local max/min/saddle pt.

Recall: In one variable, we had:

Theorem: If \( f''(x_0) = 0 \), then:

- if \( f''(x_0) > 0 \), then \( (\cup) \) \( f \) has a local min at \( x_0 \).

- if \( f''(x_0) < 0 \), then \( (\cap) \) \( f \) has a local max. at \( x_0 \).

- if \( f''(x_0) = 0 \), then the test is inconclusive.
Saddle surface
Sufficient Conditions in 2-Variable Case

Let \( f(x, y) \) such that \( f_x(a, b) = 0 = f_y(a, b) \), i.e. \((a, b)\) is a critical point of \( f \).

Consider the matrix, denoted by \( H_f \), called "Hessian" of \( f \).

\[
H_f(x, y) = \begin{bmatrix}
  f_{xx} & f_{xy} \\
  f_{yx} & f_{yy}
\end{bmatrix}
\]

We define the determinant of \( H_f(x, y) \) as:

\[
\det H_f(x, y) \overset{\text{def}}{=} f_{xx}f_{yy} - f_{xy}f_{yx}
\]

Then:

1) If \( \det H_f(a, b) > 0 \) and \( f_{xx}(a, b) > 0 \), then \( f \) has a local minimum at \((a, b)\).
2) If \( \det H_f(a, b) > 0 \) and \( f_{xx}(a, b) < 0 \), then \( f \) has a local maximum at \((a, b)\).
3) If \( \det H_f(a, b) < 0 \), then \( f \) has a saddle point at \((a, b)\); in particular, \((a, b)\) is not a local max/ min.
4) If \( \det H_f(a, b) = 0 \), then no conclusion.
Example 1

\[ f(x,y) = x^2 + y^2. \]

(Step 1 and 2)

\[ f_x = 2x, \quad f_y = 2y. \]

(0,0) is a critical point.

Prove: (0,0) is a local minimum.

Proof: Step 3:

\[ f_{xx} = 2, \quad f_{xy} = 0, \quad f_{yx} = 0, \quad f_{yy} = 2. \]

\[ H_f(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \]

\[ \text{det } H_f(x,y) = 2 \times 2 - 0 \times 0 = 4. \]

Step 4:

- \( \text{det } H_f(0,0) = 4 > 0 \)
- \( f_{xx}(0,0) = 2 > 0 \)

Hence 1) follows and (0,0) is a local minimum.
Example 2:

\[ f(x, y) = x^2 - y^2. \]

\text{(Step 1, 2)} \quad f_x = 2x, \quad f_y = -2y. \quad (0, 0) \text{ is a critical point.}

\text{Step 3: } \quad f_{xx} = 2, \quad f_{xy} = 0 = f_{yx}, \quad f_{yy} = -2

\[ \nabla^2 f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \leq \text{not a constant matrix in general.} \]

\[ \text{det} \nabla^2 f(x, y) = 2 \times (-2) - 0 \times 0 = -4 < 0. \]

\text{Step 4: } \quad \text{det} \nabla^2 f(0, 0) = -4 < 0 \quad \text{to substitute (0, 0) into the formula.}

Hence \( f \) has a saddle point at (0, 0).