Learning Goals for Today:

1. Functions of several variables (2 in most cases)
2. Partial Differentiation.

E.g.

\[ 2x + 3y + z = 1 \quad \text{(Plane)} \]

Rewrite it as

\[ z = 1 - 2x - 3y \]

If we denote

\[ f(x, y) = 1 - 2x - 3y \]

Then:

\[ f \] is a function of

2 independent variables

\( x \) and \( y \).

Recall: Derivatives of functions of one variable,

- \( x_1 \): (flat, \( f'(x_1) \) is small)
- \( x_2 \): steep, \( f'(x_2) \) is large.

Grouse Mountain (Different slopes)
e.g. \( \begin{cases} \quad x^2 + y^2 + z^2 = 1 \\ z \geq 0 \end{cases} \) 

\[ z = \sqrt{1 - x^2 - y^2} \]

The upper half of the unit sphere centred at the origin.

**Domain of multivariable functions**

Natural domain of multivariable functions: the largest set of \((x, y)\)'s for which \(f(x, y)\) makes sense.

- **Example 1:** \(f(x, y) = 1 - 2x - 3y\)  
  Domain: \(\mathbb{R}^2\)  
  \(\mathbb{R} = \{ (x, y) : x \in \mathbb{R}, y \in \mathbb{R} \}\)

- **Example 2:** \(g(x, y) = \sqrt{1 - x^2 - y^2}\)  
  Natural domain: \(1 - x^2 - y^2 \geq 0\)  
  \(\mathbb{R} \setminus \{(0, 0)\}\)

- **Example 3:** \(h(x, y) = \log(xy)\)  
  Natural domain: \(xy > 0\)

The unit disk (closed i.e. with boundary)
Partial Differentiation

Definition: Given $f(x, y)$, fix $(a, b)$ in the domain of $f(x, y)$. The partial derivative of $f$ with respect to $x$ at $(a, b)$ is defined by:

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

Notations on LHS: $\frac{\partial f}{\partial x}$ (instead of $\frac{df}{dx}$)

pronunciation: "del $f$ del $x$" or "dee $f$ dee $x$"
The moral of partial differentiation.

To compute \( \frac{\partial f}{\partial y} \frac{df}{dx} (x, y) \), just differentiate w.r.t. \( x \), treating \( y \) as a constant.

\[ f(x, y) = x^2 y \]

Find \( \frac{\partial f}{\partial x} (x, y) \), \( \frac{\partial f}{\partial y} (x, y) \).

\[ \frac{\partial f}{\partial x} (x, y) = \frac{\partial}{\partial x} (x^2 y) = 2xy \]

\[ \frac{\partial f}{\partial y} (x, y) = \frac{\partial}{\partial y} (x^2 y) = x^2 \]

\[ \frac{df}{dx} (x, y) \text{ as "constant"} \]

Good notation "constant"
Recall: \( y = f(x) \)

Slope at \( x_0 = f'(x_0) = \text{the rate of change of } f \text{ at } x_0 \). (as \( x \) increases)

In 2-D case,

\[
\frac{df}{dx}(a,b) = \text{the rate of change of } f \text{ along the positive } x\text{-axis at } (a,b)
\]

\[
\frac{df}{dy}(a,b) = \text{the rate of change of } f \text{ along the positive } y\text{-axis at } (a,b)
\]
Exercise:
Let \( f(x,y) = \frac{x}{x^2 + y^2} \).

Question:  
1. Find the natural domain of \( f \).
2. Within the natural domain of \( f \), find \( \frac{\partial f}{\partial x}(x,y) \) and \( \frac{\partial f}{\partial y}(x,y) \).

Higher Order Derivatives

Differ \( \frac{\partial}{\partial x} \). Analogue to \( f'' \), \( f''' \) in the single variable case.

If we differentiate \( f \) twice, what are the possibilities?

\[ \frac{\partial^2 f}{\partial x^2} = f_{xx} \]
\[ \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \]
\[ \frac{\partial^2 f}{\partial y^2} = f_{yy} \]

\((\text{Denoted})\)

\[ \frac{\partial (\frac{\partial f}{\partial x})}{\partial x} \]
\[ \frac{\partial (\frac{\partial f}{\partial y})}{\partial y} \]

4: \( \frac{\partial (\frac{\partial f}{\partial x})}{\partial x} \) \( \frac{\partial (\frac{\partial f}{\partial y})}{\partial y} \)