Recall:

Last Time: Equation of Planes, Relations between Planes.

This Time: Distance between Planes
- Distance between point and a plane.

Theorem 1: Let \( P_1 : ax + by + cz = d_1 \) be planes
\( P_2 : ax + by + cz = d_2 \)

Then the distance between \( P_1 \) and \( P_2 \) is given by

\[
\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}
\]

Example 1: Let \( P_1 : x + y + z = 1 \) \( \vec{n}_1 = <1, 1, 1> \)
\( P_2 : -x - y - z = \frac{1}{2} \) \( \vec{n}_2 = <-1, -1, -1> \)
Find the distance between \( P_1 \) and \( P_2 \).

Solution: Step 1: Write Rewrite \( P_2 \): (Multiplying \((-1)\))
\( x + y + z = -\frac{1}{2} \)
\[
\text{Then distance} = \left| 1 - \left(-\frac{1}{2}\right) \sqrt{1^2 + 1^2 + 1^2} \right| = \frac{3}{2}/\sqrt{3}
\]
Next: Distance between points and planes

\[ P (x_0, y_0, z_0) \]

\[ a : ax + by + cz = d \]

How do we find distance between \( P \) and \( \alpha \)?

Sol: Step 1: Draw (find) the equation of plane // to \( \alpha \), and passing through \( P \). (denoted \( \beta \))

e.g. Find distance between \( \beta \) and \( x+y+z=1 \).

Then Step 1:

\[ \vec{n} = <1, 1, 1> \]

We know \( \beta \) passes through the origin \((0, 0, 0)\).

The equation for \( \beta \):

\[ 1 \cdot (x-0) + 1 \cdot (y-0) + 1 \cdot (z-0) = 0 \]

\[ \Rightarrow x + y + z = 0 \quad \text{(eqn of } \beta) \]

\[ \Rightarrow \text{Distance between } \alpha \text{ and } \beta = \frac{|1 - 0|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}} \]
Surfaces in $\mathbb{R}^3$.

What is a surface?

A surface in $\mathbb{R}^3$ is the set of all points $(x, y, z)$ satisfying an equation of the form

$$f(x, y, z) = 0,$$

(*)

where $f$ is a "good" function.

e.g. 1) $f(x, y, z) = 2x + 3y - z - 1$.
    Then $\& 2x + 3y - z - 1 = 0$ is the eqn. of a plane.

2) $f(x, y, z) = z$.
    Then $z = 0$ gives the $xOy$-plane.

3) $f(x, y, z) = x^2 + y^2 + z^2 - 1$.

Q: What does it look like?

Ans: Setting $f(x, y, z) = 0$ as in (*)

$$x^2 + y^2 + z^2 = 1$$

a sphere (a unit sphere centred at 0)
Trace: By def. Trace is the intersection of a surface with a plane given by eqns of the form $x = \text{constant}$, $y = \text{constant}$ OR $z = \text{const.}$

E.g. $f(x, y, z) = x^2 + y^2 - z = 0$. (***)

Q: Find the trace of the surface given by (***) at $x = 0, 1, \text{ and } 2$.

Sol: From (***)

$z = x^2 + y^2$

If $x = 0$, $z = y^2$;
If $x = 1$, $z = y^2 + 1$;
If $x = 2$, $z = y^2 + 4$.

Ex: Find the trace of the surface given by (***) at $y = -1, 0, \text{ and } 1.$
Contour (Level Sets)

A level set is the trace of a surface by setting $z = \text{constant}$.

Example.

Find the level sets of

$$z = \sqrt{x^2 + y^2}$$

at $z = 0, 1, 2$ respectively.

Solution.

In 3-D

If $z = 0$,

$$\sqrt{x^2 + y^2} = 0$$

$\implies x = y = 0$

If $z = 1$,

$$\sqrt{x^2 + y^2} = 1$$

$\implies x^2 + y^2 = 1$

If $z = 2$

$$\sqrt{x^2 + y^2} = 2$$

$\implies x^2 + y^2 = 4$
Some Softwares:

Wolfram Alpha, Geogebra (Easier to use)
Matlab, Mathematica (Requires some basic knowledge of computing languages)

Examples of surfaces: (For reference, not covered in lectures, also available online)

1. \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \): Ellipsoid

2. \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \): Hyperboloid of 2 sheets

3. \( \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \): Hyperboloid of 1 sheet