Area Function: \( f \) is continuous
\[
F(x) = \int_a^x f(t) \, dt.
\]

Want: \( F'(x) = f(x) \).

Sol: By definition of derivative,
\[
F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}
\]
\[
= \lim_{h \to 0} \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h}
\]
\[
= \lim_{h \to 0} \frac{\int_x^{x+h} f(t) \, dt}{h}
\]
\[
\approx \lim_{h \to 0} \frac{(x+h-x) \cdot f(x)}{h}
\]
\[
= f(x).
\]
Example 1: If $F(x) = \int_x^{x^2} \cos(t) \, dt$.

Find $F'(x)$.  

Sol: Method 1:

Using FTC, version II.

That is, we find $\int_a^b \cos(t) \, dt = G(b) - G(a)$, where $G'(t) = \cos(t)$.

A natural choice is $G(t) = \sin(t)$.

Hence $\int_a^b \cos(t) \, dt = \sin(b) - \sin(a)$.

Hence if we let $b = x^2$, $a = x$,

$F(x) = \sin(x^2) - \sin(x)$.

$\Rightarrow F'(x) = \cos(x^2) \cdot (2x) - \cos(x)$. 
Method 2:
Using FTC, Version I.
Write \( f(t) = \cos(t) \).
Write \( G(x) \) to be the area function defined by \( f(x) \), i.e.
\[
G(x) = \int_0^x \cos(t) \, dt.
\]
Choose 2 different letters.

\( x \) can be chosen to be any number within the domain of \( \cos(t) \).

Then, by FTC, Version I,
\[
G'(x) = \cos(x). \quad (*)
\]

Now \( F(x) = \int_x^{x^2} \cos(t) \, dt \).
(By Version II of FTC)
\[
F'(x) = G'(x^2) \cdot 2x - G'(x) \quad \text{(*) = } \cos(x^2) \cdot 2x - \cos(x).
Example 2.

If \( F(x) = \int_{\ln(x)}^{1} e^{\cos(t)} \, dt \), find \( F'(x) \).

Sol. (Method 1 fails as we don't know any antiderivative of \( e^{\cos(t)} \)).

Using Method 2.

Define \( G(x) = \int_{0}^{x} e^{\cos(t)} \, dt \).

Then by FTC, version I,

\[
G'(x) = e^{\cos(x)} \tag{1}
\]

Then \( F(x) = G(1) - G(\ln(x)) \).

(By FTC, version II)

\[
\Rightarrow F'(x) = 0 - G'(\ln(x)) \cdot \frac{1}{x} - \cos(\ln(x)) \cdot \frac{1}{x}
\]
Going back to FTC, version II, and use it to find integrals.

Recall: Let $F' = f$, say, on $[a, b]$, (continuous)
then
$$\int_{a}^{b} f(t) \, dt = F(b) - F(a).$$

Two steps to find $\int_{a}^{b} f(t) \, dt$.

Step 1: Find an antiderivative $F$ of $f$. (Difficult part)

Step 2: Evaluate $F(b) - F(a)$.

Today: Antiderivatives of some common functions.
Notation: \(\int f(t)\,dt\) (Indefinite Integral)  

We write \(\int f(t)\,dt = F(t) + C\)  

(For comparison, \(\int_{a}^{b} f(t)\,dt\) is called a definite integral)  

No upper limit. \(\int f(t)\,dt\) is nothing lower limit \(\int_{0}^{0} f(t)\,dt\) is nothing  

if \(F' = f\).  

\(e.g.\ \int 1\,dx = x + C\)  

\(\int x\,dx = \frac{1}{2} x^2 + C\)  

\(\int t\,dt = \frac{1}{2} t^2 + C\)  

\(\int y\,dy = \frac{1}{2} y^2 + C\)  

As long as these are the same, they mean the same integral.
Table of antiderivatives (indefinite integrals) for some elementary functions.

<table>
<thead>
<tr>
<th>Function</th>
<th>Indefinite Integrals</th>
<th>Derivative (For comparison)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x + C$</td>
<td>0</td>
</tr>
<tr>
<td>$x^n$ $(n \neq -1)$</td>
<td>$\frac{x^{n+1}}{n+1} + C$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$x^{-1}$</td>
<td>$\ln(x) + C$</td>
<td>$-x^{-2}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x + C$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$-\cos(x) + C$</td>
<td>$\cos(x)$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\sin(x) + C$</td>
<td>$-\sin(x)$</td>
</tr>
<tr>
<td>$\sec^2(x)$</td>
<td>$\tan(x) + C$</td>
<td>Exercise.</td>
</tr>
<tr>
<td>$(1-x^2)^{-\frac{1}{2}}$</td>
<td>$\arcsin(x) + C$</td>
<td>Exercise.</td>
</tr>
</tbody>
</table>
Want: $x^n$?

Recall:

$$\left(x^m\right)' = mx^{m-1} \quad (*)$$

Set $n = m - 1$, so $m = n + 1$.

Hence $(*)$ becomes:

$$\left(x^{n+1}\right)' = (n+1)x^n.$$  

- If $n+1 \neq 0$, then

  $$\left(\frac{x^{n+1}}{n+1}\right)' = x^n.$$  

- If $n+1 = 0$, i.e. $n = -1$,

  recall $\left(\log(x)\right)' = \frac{1}{x} = x^{-1}.$

Recall:

$$\left(\arcsin(x)\right)' = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$\left(\tan(x)\right)' = \sec^2(x) = \frac{1}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)^2.$$
Example 1:

Find \( \int_{1}^{2} \frac{1}{\sqrt{t}} \, dt \).

Solution: Step 1: Write \( f(t) = \frac{1}{\sqrt{t}} = t^{-\frac{1}{2}} \).

Then \( F(t) = \frac{t^{-\frac{1}{2} + 1}}{-\frac{1}{2} + 1} + C \)

\[ = 2 \, t^{\frac{1}{2}} + C \]

Any choice of \( C \) does not affect step 2.

Step 2:

\( \int_{1}^{2} \frac{1}{\sqrt{t}} \, dt = F(2) - F(1) \)

\[ = 2 \times 2^{\frac{1}{2}} - 2 \times 1^{\frac{1}{2}} \]

\[ = 2 \sqrt{2} - 2 \]
Example 2:

Find \( \int_0^1 x(x-2) \, dx \).

Sol:

\[
\begin{align*}
\int_0^1 x(x-2) \, dx &= \int_0^1 x^2 - 2x \, dx \\
&= \left[ \frac{1}{3} x^3 - x^2 \right]_0^1 \\
&= \left[ \frac{1}{3} - 1 \right] - \left[ 0 - 0 \right] \\
&= \frac{1}{3} - 1 = -\frac{2}{3}.
\end{align*}
\]