Announcement:

1. Both Webworks 10 and 11 due on Saturday, April 6th, at 11 pm.
2. The worst Webwork will be dropped (for all sections).
3. Sample final questions available on the Common Webpage.
4. The Math Learning Centre is open starting from April 5th until April 12th.
5. You can get your quiz back from the MLC starting from Friday.
Outline for today:

1. Differentiation and integration of power series.

2. Computation of Taylor (Maclaurin) Series.

Theorem (Termwise Differentiation)

Given a power series \( \sum_{n=0}^{\infty} a_n (x-a)^n \)
which is convergent on some interval \( I \).

Then we have, for all \( x \in I \),

\[
\left( \sum_{n=0}^{\infty} a_n (x-a)^n \right)' = \sum_{n=1}^{\infty} a_n \cdot n (x-a)^{n-1}.
\]

and \( \int \sum_{n=0}^{\infty} a_n (x-a)^n \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1} + C. \)
E.g. Find the power series representation of the function \( f(x) = \arctan(x) \) at \( a = 0 \). (Notation: sometimes people write \( x = 0 \) to mean that \( a = \text{centre} = 0 \).

Sol: This question asks us to find a power series with \( a = 0 \), such that

\[
f(x) = \arctan(x) = \sum_{n=0}^{\infty} a_n x^n.
\]

Let us observe that

\[
(\arctan(x))' = \frac{1}{1+x^2}
\]

Let us observe that \((\text{Last time's notes})\)

\[
\frac{1}{1+x^2} = \frac{1}{1-(x^2)} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + \ldots
\]

\[
= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots
\]

for \( |x| < 1 \). \( (\ast) \)
Integration on both sides (to recover \( \arctan(x) \)) of (*) from \( \frac{1}{1+x^2} \):

\[
\arctan(x) = \int \left(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots\right) \, dx
\]

(by Thm just now) \( = C + x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \frac{1}{9} x^9 + \ldots \)

(Note: We have to find \( C !! \))

(For example, taking \( x = 0 \), \( \text{LHS} = 0 \), \( \text{RHS} = C \).
This shows that \( C = 0 \).) (Recall: Similar to solving a differential equation)

The final answer:

\[
\arctan(x) = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \frac{1}{9} x^9 + \ldots
\]

(Take e.g. \( x = \frac{\pi}{4} \), \( \text{LHS} = \frac{\pi}{4} \), \( \text{RHS} \)
\[
x = 1
\]
\[
\text{RHS} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots
\]
\[
\Rightarrow \pi^4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots
\]

**Taylor (Maclaurin) Series**

Recall: (In Differential Calculus)

Given a good function \( f \), the \( n \)-th Taylor polynomial of \( f \) is given by

\[
T_n(f) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

\[
= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{f^{(4)}(a)}{24}(x-a)^4 + \ldots
\]

\[
= \frac{f^{(n)}(a)}{n!} (x-a)^n.
\]

(Approximation of \( f \) by \( T_n(f) \))
Notice that:

- $T_n(f)$ is a polynomial, not a series.
- As $n \to \infty$, it is expected that $T_n(f)$ would be close enough to $f$,

$$\Rightarrow \quad T_n(f) \to f \quad \text{as} \quad n \to \infty$$

(convergent)

$$\Rightarrow \quad f(x) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(x-a)^k}{k!}.$$

(Hence we can replace the \(n\) \(\in\) in (**) by \(\infty\), in good situations) (i.e., the series converges.

Definition: Given a function $f$. Then the Taylor series of $f$ around the centre $x = a$ is given by

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(x-a)^k}{k!}.$$
If \( a = 0 \), we say the Taylor series is a Maclaurin series.

Example 2. Find the Maclaurin series for \( f(x) = e^x \).

Sol: By definition, if \( f(x) = e^x \), then 
\[
\begin{align*}
    f'(x) &= e^x, \\
    f''(x) &= e^x, \\
    \vdots \\
    f^{(k)}(x) &= e^x.
\end{align*}
\]

Hence for all \( k \geq 0 \), 
\[
\frac{f^{(k)}(0)}{k!} x^k = e^0 / x^0 = 1.
\]

By the formula of Taylor series,
\[
f(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.
\]

\[\Box\]
Example 3. Find the Maclaurin series for $f(x) = e^{x^2}$.

Sol: (2 ways.
First way: to use the definition, and find the $n$-th derivative of $f(x)$ (using chain rule for a general $n$).
(This is almost an impossible task!)

Second way: To use a little trick.
Let $u = x^2$.
Then by Example 2,

$$f(x) = e^{x^2} = e^u = \sum_{k=0}^{\infty} \frac{1}{k!} u^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$$
Example 4:

Find the Maclaurin Series for $f(x)$ at $x = 0$, where

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

(Recall: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, so $f$ is continuous at $x = 0$)

Sol. (Don't try to differentiate $\frac{\sin(x)}{x}$ n-times)

Instead, we notice that (Exercise by definition of Taylor Series)

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= x - \frac{1}{6} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots$$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
\[
\frac{\sin(x)}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}
\]

\[
= 1 - \frac{1}{6} x^2 + \frac{1}{5!} x^4 - \frac{1}{7!} x^6 + \ldots
\]