Lecture 24

Example:

\[ f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1 \]

\[ f'(x) = 12(x-1)^2(x+1) \]

Q: Find critical points and classify them.

- \( x = 1 \), \( x = -1 \) are critical points.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign ( f' )</th>
<th>((x-1)^2)</th>
<th>( x+1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -1))</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>((1, \infty))</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>
Therefore by 1st Derivative Test:

-1 is a local min
1 is not a min or max

Ex: \( f(x) = \sqrt{x} \ln(x) \)

Q: Find local min/max using 1st Derivative test
   - Find absolute max/min (on the domain of f)
SoC: \[ f'(x) = \frac{1}{2\sqrt{x}} \ln(x) + \sqrt{x} \cdot \frac{1}{x} \]

\[ = \frac{1}{2} \sqrt{x} \ln(x) + \frac{1}{\sqrt{x}} \]

\[ = \frac{1}{\sqrt{x}} \left( \frac{1}{2} \ln(x) + 1 \right) \]

\[ = 0 \]

\[ \frac{1}{\sqrt{x}} \neq 0 \text{ so } \frac{1}{2} \ln(x) + 1 = 0 \]

\[ \ln(x) = -2 \]

\[ x = e^{-2} \]

So, \( e^{-2} \) is the only critical point.

Check this using 1st derivative test.
So one needs to know if

\[ f'(x) = \frac{1}{\sqrt{x}} \left( \frac{1}{2} \ln(x) + 1 \right) \]

changes sign at \( x = e^{-2} \).

Here: \( \frac{1}{\sqrt{x}} \) is +

\[ \frac{1}{2} \ln(x) + 1 > -2 \]

\[ \ln(x) > 2 - 1 \]

\[ \ln(x) > 1 \]

\[ f(x) = \ln(x) \]
\[ x > e^{-2} \]
\[ \ln(x) > -2 \quad \text{so} \quad \frac{1}{2} \ln(x) + 1 > 0 \]

Hence:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign of ( f' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, e^{-2}))</td>
<td>-</td>
</tr>
<tr>
<td>((e^{-2}, \infty))</td>
<td>+</td>
</tr>
</tbody>
</table>

Alternatively, plug in values:

\[ x < e^{-2} : \quad \ln x = e^{-3} \]
\[ f'(e^{-3}) = \frac{1}{\sqrt{e^{-3}}} \cdot \left( \frac{1}{2} (-3) + 1 \right) < 0 \]

\[ x > e^{-2} : \quad x = e \quad \text{or} \quad x = 1, \quad x = e^{-1} \]
\[ f'(e) = \frac{1}{\sqrt{e}} \left( \frac{1}{2} \cdot 1 + 1 \right) > 0 \]
Thus: \( L' \) changes from \(-\) to \(+\)

\[ a \rightarrow x = e^{-2} \]

so \( e^{-2} \) is local min.

What about (absolute max)/min?

\[ f(e^{-2}) = \frac{\sqrt{e^2}}{e} m(e^{-2}) = \frac{1}{e} (e) \]

So: because \( L \) decreases for all \( x < e^{-2} \)

\[ \frac{\text{increases for}}{\text{all} \ x > e^{-2}} \]
$x = e^{-2}$ is also absolute min of $f(x)$.

Local min is also absolute min.

Local max is also absolute max.
FACT: If a function $f(x)$ has exactly one local max (or local min) and no other critical points, it is necessarily an absolute max (or absolute min).

Note: This is not true if you have more than 1 C.P.
Ex: Show \( f(x) = x^x \) has an absolute extreme value on \((0, \infty)\).

Sol: Want to show exactly one local extreme value:

\[
\frac{d}{dx} (x^x) = (e^{\ln(x^x)})' = x \cdot \ln(x)(x \ln(x))'
\]

(or logarithmic differentiation)

\[
= x^x \left( \ln(x) + x \cdot \frac{1}{x} \right) = x^x \left( 1 + \ln(x) \right)
\]
\( f'(x) = 0 \)

\[ x^x \left( 1 + \ln(x) \right) = 0 \]

So \( 1 + \ln(x) = 0 \) as \( x \neq 0 \)

\[ \ln(x) = -1 \]

\[ x = e^{-1} \]

Want to show \( x = e^{-1} \) is a local extreme value:

1st Derivative Test

\[ f'(x) = x^x \left( 1 + \ln(x) \right) \]

\( x > e^{-1} \)  \( \ln(x) > -1 \)  \( f'(x) > 0 \)

\( x < e^{-1} \)  \( \ln(x) < -1 \)  \( f'(x) < 0 \)
So: $f'$ changes from $-\text{ to } +$ at $x = e^{-1}$.

It is a local min.

Hence, by previous fact (i.e. only one local extremum),

$x = e^{-1}$ is absolute min of $f(x) = x^+$.
The second derivative

\[ f' > 0 \quad f' < 0 \]

For second derivative:

\[ f'' > 0 \quad f'' < 0 \]

\[ f(x) = x^2 \]
\[ f'(x) = 2x \]
\[ f''(x) = 2 \]

\[ f(x) = -x \]

\[ f'' > 0 \]
Call $d'' > 0$ concave up

$d'' < 0$ concave down

Concave up: $f(x)$ conc. up,

$d''(x) > 0$, so $f'(x) > 0$.

Increasing: Slopes of tangent lines increase
Concave down:

$\delta$ to $\gamma$ the scopes of tangent lines decrease
Definition:

Say \( f \) is twice differentiable,

1. \( f'' > 0 \) on some interval \( I \), \( f \) is concave up on \( I \)
2. \( f'' < 0 \) on some interval \( I \), \( f \) is concave down on \( I \)

(Remember graphs of \( x^2 \) and \(-x^2\))
Finding Intervals of Concavity

$f(x) = x^3$

Concave down

Concave up

Inlection point

$(x^3$ changes concavity)

<table>
<thead>
<tr>
<th>Increasing</th>
<th>Decreasing</th>
</tr>
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<tbody>
<tr>
<td>$\rightarrow$</td>
<td>$\leftarrow$</td>
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</table>

"Switching"
<table>
<thead>
<tr>
<th>Concave up</th>
<th>Increasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concave down</td>
<td>Decreasing</td>
</tr>
<tr>
<td>Inflection point</td>
<td>&quot;Switches&quot;</td>
</tr>
<tr>
<td>= Change of Sign</td>
<td></td>
</tr>
<tr>
<td>1'</td>
<td>+</td>
</tr>
<tr>
<td>1''</td>
<td>-</td>
</tr>
<tr>
<td>0 or DNE</td>
<td></td>
</tr>
<tr>
<td>(C.P. of 1'')</td>
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