Lecture 22

Maxima and Minima of Functions

(Chapter 4.11)

Absolute vs. Local

Derivative and Local Max/Min:

Fact: If \( f(x) \) is differentiable at \( x = c \) and \( c \) is local extremum, \( f'(c) = 0 \).
Ex: \( f(x) = x^3 - x \)

Q: What are the local extrema of \( f \)?

SoC: \( f'(x) = 0 \)

\[ 3x^2 - 1 = 0 \]
\[ x^2 = \frac{1}{3} \]
\[ x = \pm \sqrt[3]{\frac{1}{3}} \]

We can say: Only possible values of \( x \) are \( \pm \sqrt[3]{\frac{1}{3}} \)

but no more.
Ex: \[ f(x) = x^3 \]

Here: \[ f'(x) = 3x^2 = 0 \]

So: \[ f'(x) = 0 \text{ at } x = 0 \]

But: from graph, we see 0 is not a local max/min.
Ex: \( f(x) = |x| \)

We can see: 0 is a local min.

But: \( f'(0) \) DNE

("corner" at \( x = 0 \))

Ex: \( f(x) \) local max

\( f'(0) \) DNE
Critical Points

Def: A point $c$ in the interior of the domain of $f$ is called critical if:

- $f'(c) = 0$
- or $f''(c)$ DNE

Ex: $f(x) = x^3 - x$

We saw $\pm \sqrt{\frac{1}{3}}$ is critical

$\Rightarrow (x) = x^3$

$x = 0$ is critical
Ex: Find all critical points of \( f(x) = x^2 \ln(x) \).

SoC: \( x > 0 \) is the domain.

\[
\begin{align*}
\frac{d}{dx} f(x) &= 2x \ln(x) + \frac{x^2}{x} \\
&= 2x \ln(x) + x \\
&= x(1 + 2 \ln(x)) \\
&= 0
\end{align*}
\]

Either \( x = 0 \) (not possible) or \( 1 + 2 \ln(x) = 0 \) so \( 1 + 2 \ln(x) = 0 \).

\[
\ln(x) = -\frac{1}{2}
\]

\[
x = e^{-\frac{1}{2}}
\]
Hence: $x = e^{-\frac{1}{2}}$ is the only critical point.

Critical points vs Local Extrema

"Local Extremum" implies "critical point"
Ex:

Critical, not an extremum

Ex:

Critical, not an extremum
Ex: \( f(x) = 12x^5 - 20x^3 \)

Q: Find all critical points.

Soc: \( f'(x) = 60x^4 - 60x^2 \)

\[ = 60x^2(x^2 - 1) \]

\[ = 60x^2(x+1)(x-1) \]

\[ = 0 \]

\( x = 0 \) \( x = \pm 1 \) critical points

Remark: To find critical points, try and factor \( f' \) as much as possible.
Finding absolute max/min on a closed interval

Observation: if \( c \in (a, b) \) 
\( (c \neq a \text{ or } c \neq b) \)

\( f(c) \) absolute max/min
\( \Rightarrow f'(c) \) local max/min
\( \Rightarrow c \) is a critical point
Ex: \( f(x) = x^4 - 2x^3 \)

Q: Absolute max/min on \([-2, 2]\).

SoC: \( f'(x) = 4x^3 - 6x^2 = 0 \)

\[ = x(4x^2 - 6x) = 0 \]
\[ = x^2(4x - 6) = 0 \]

\( x = 0 \quad 4x = 6 \quad \frac{3}{2} \)

are critical points in \([-2, 2]\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^4 - 2x^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>[ 32 ] max</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>( (\frac{3}{2})^4 - 2 \cdot (\frac{3}{2})^3 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
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</table>
\[ (\frac{3}{2})^4 - 2 \cdot (\frac{3}{2})^3 \]
\[ = (\frac{3}{2})^3 \cdot (\frac{3}{2} - 2) \]
\[ = -\frac{1}{2} (\frac{3}{2})^3 < 0 \]

So: \[-\frac{1}{2} (\frac{3}{2})^3\] is \text{min} at \( x = \frac{3}{2} \)

\[ 3.2 \] is \text{max} at \( x = -2 \)

\[ \text{In general:} \]

\[ \text{Max (Min of } f \text{ on } [a,b]) \]

1) Find critical points \( c \text{ in } (a,b) \)
2) Compute \( f(a), f(b) \)
3) Find largest/smallest from 1), 2).
Ex: \( f(x) = x^3 e^{-x} \)

**Absolute max/min on \([-1, 5]\)**

SoC: \( f'(x) = 3x^2 e^{-x} - x^3 e^{-x} \)

\[ = e^{-x} (3x^2 - x^3) \]

\[ = x^2 e^{-x} (3 - x) \]

So \( x = 0 \) and \( x = 3 \) are critical points, both in \([-1, 5]\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^3 e^{-x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>(-e)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>27 e^{-3}</td>
</tr>
<tr>
<td>5</td>
<td>5^3 e^{-5} = 125 e^{-5}</td>
</tr>
</tbody>
</table>
Q: $3^5 \; 3^3 \; e^{-3}$ or $5^3 \; e^{-5}$ \[ \hat{\text{bigger or smaller?}} \]

To see this: Find whether $f'$ or $f''$ between $3, 5$:

$$f'(x) = x^2 e^{-x} (3-x) > 0$$

\[ \uparrow \uparrow \uparrow \uparrow \uparrow \]

\[ x > 3 \]
On \( [3, 5] \), so:

\[
3^3 e^{-3} > 5^3 e^{-5}
\]

Conclusion:

\[ x = -1 \text{ is } \min \]

\[ x = 3 \text{ is } \max \]
Ex: \( f(x) = x^{\frac{2}{3}} (2-x) \)

Find absolute max/min on \((-1, 7]\).

SoC: \( f'(x) = \frac{2}{3} x^{-\frac{1}{3}} (2-x) + x^{\frac{2}{3}} (-1) \)

Q: When does \( f \) exist?

\[ \text{always} \]

Q: When does \( f' \) exist?

\[ x = 0 \] & \( f' \) DNE